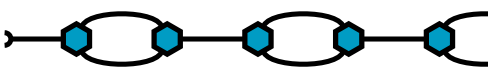
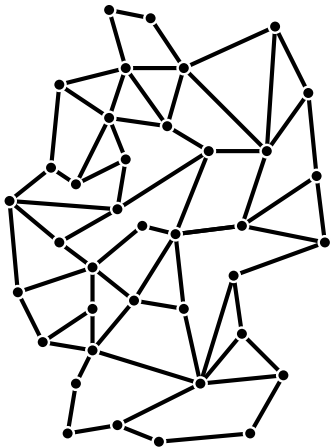
Faster, Higher, Easier: Toward a Systematic Study of Parameterized Vertex and Edge Selection Problems

Philipp Schepper



September 29, 2025

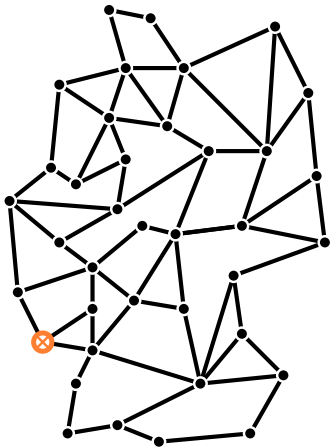
Fire Protection Train (FPT) Problem



Fire Protection Train (FPT) Problem

Task:

Find (few) locations to position the fire trains at.



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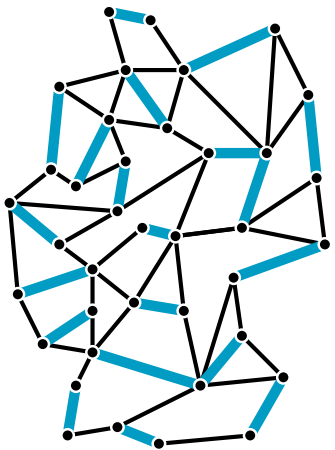

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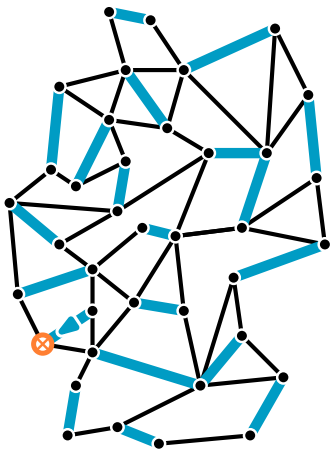

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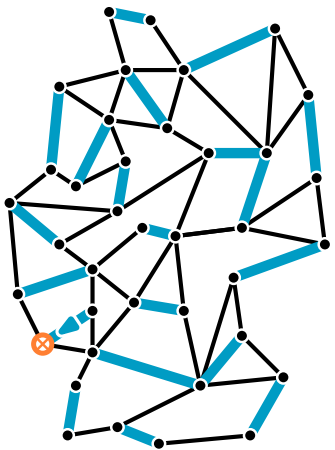
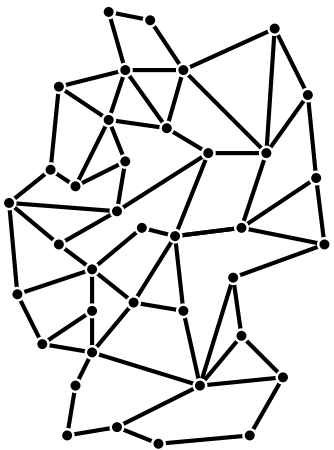


Stationary Position



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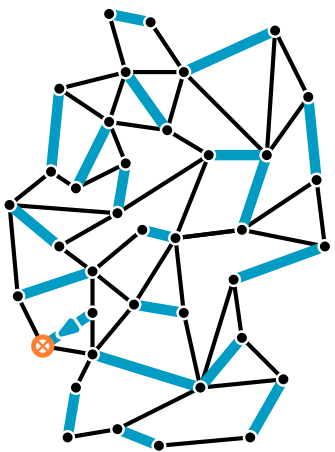

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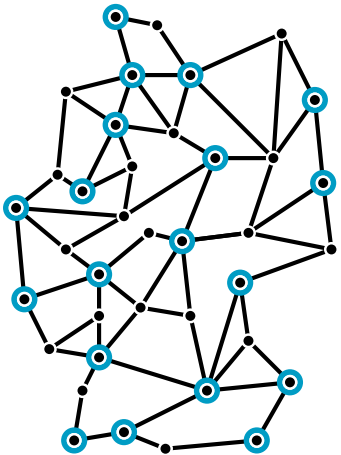

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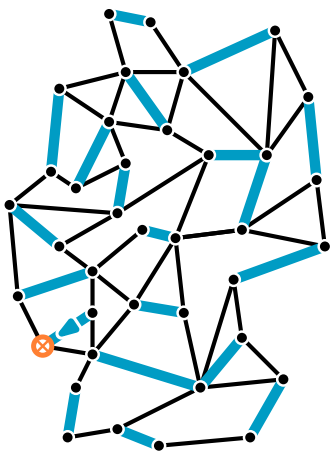

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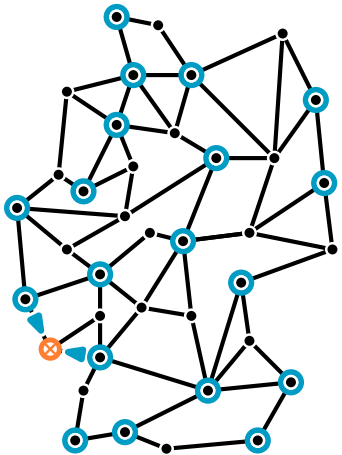


Fire Protection Train (FPT) Problem

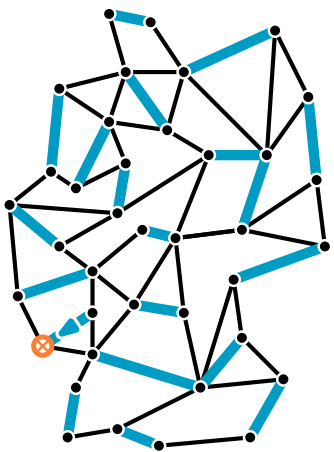
Task:

Find (few) locations to position the fire trains at.

Next:

Formalize and solve the problem!

Moving Train



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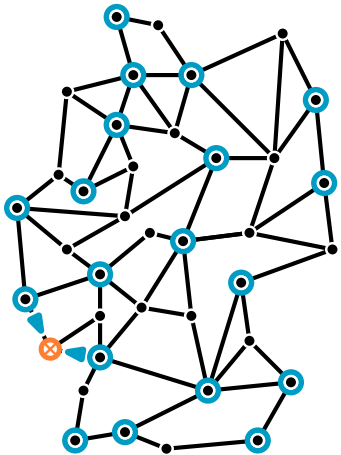
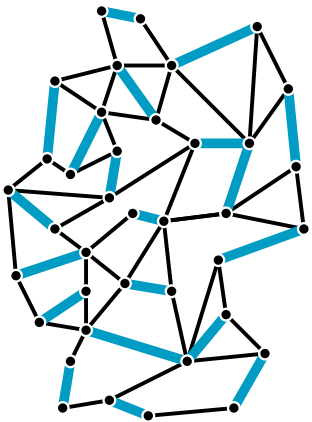
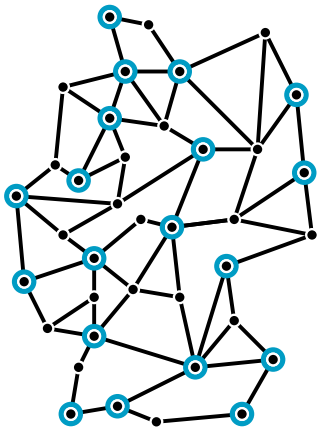


Fire Protection Train (FPT) Problem: Formalization





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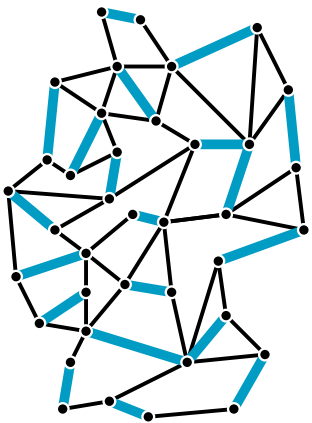
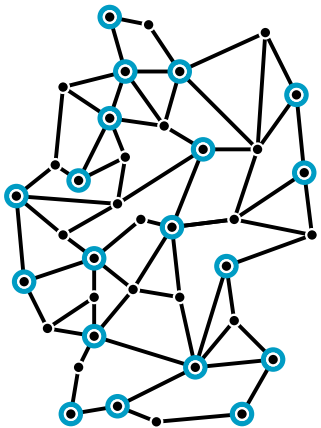


Edge Cover

Each vertex incident to ≥ 1 *selected edge*.



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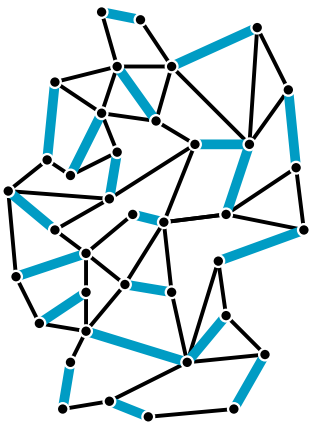

Edge Cover

Each vertex incident to

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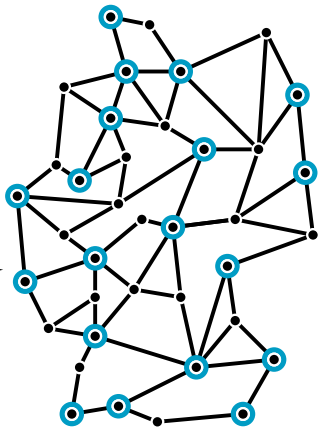
Total 2-Domination

Selected vertex:

≥ 1 selected neighbor.

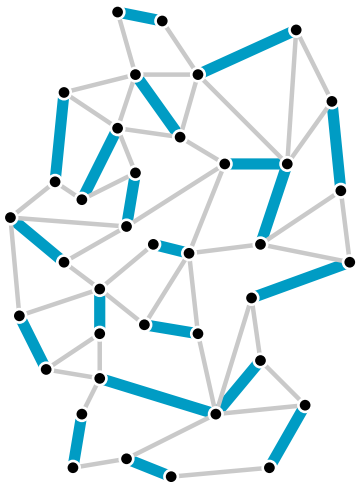
Unselected vertex:

≥ 2 selected neighbor.



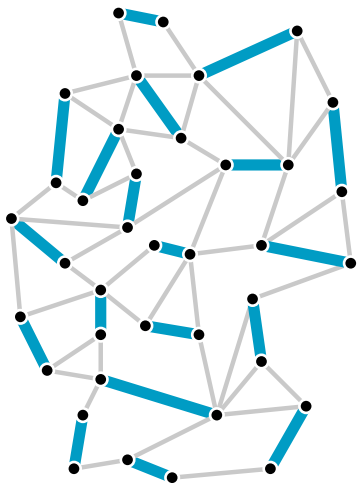
More Edge and Vertex Selection Problems

	Selected	Unselected
Edge Cover	≥ 1	



More Edge and Vertex Selection Problems

	Selected	Unselected
Edge Cover	≥ 1	
Matching	≤ 1	
Perfect Matching	$= 1$	



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Edge Cover	≥ 1	
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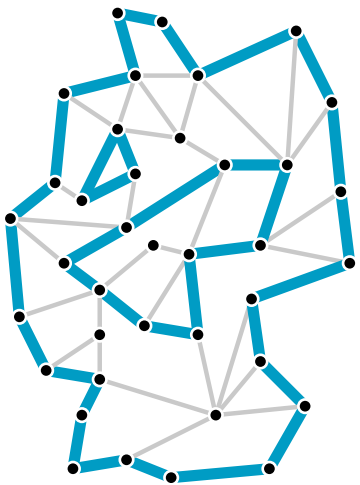
Cycle Packing	$= 0$ or $= 2$	



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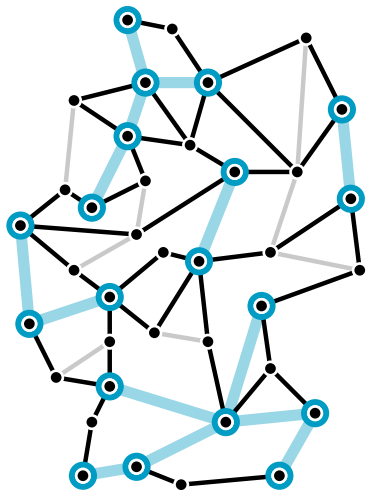

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Edge Cover	≥ 1	
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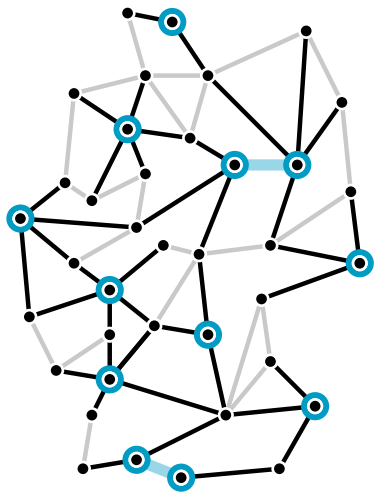
Total 2-Domination	≥ 1	≥ 2
Dominating Set	≥ 0	≥ 1



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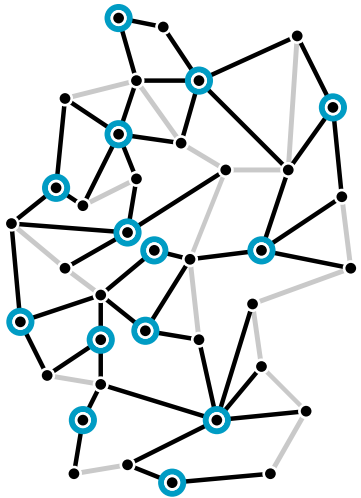
Independent Set	$= 0$	≥ 0



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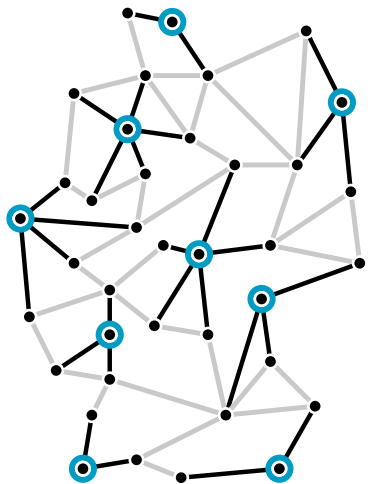

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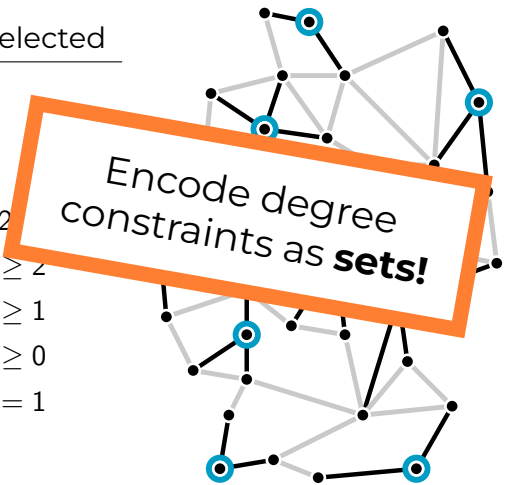

Selecting Edges → Generalized Factor

Fix a set $\Lambda \subseteq \mathbb{N}$.

Λ-Factor [Lovász 1972]

Given: A simple graph G.

Is there an *edge set* $S \subseteq E(G)$ such that
for every vertex $v \in V(G)$ we have $\deg_S(v) \in \Lambda$?

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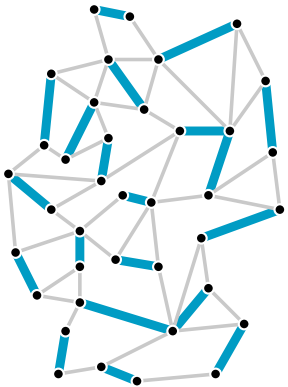

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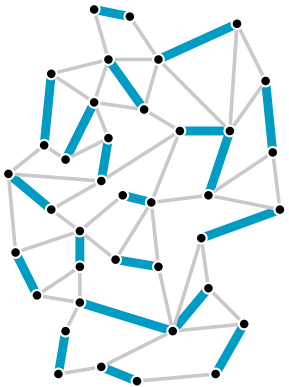

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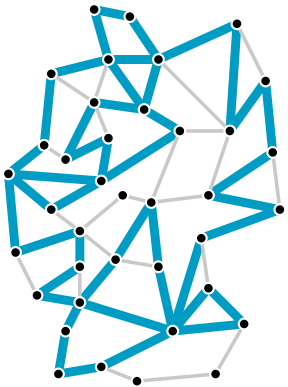

$\{0, 2, 4, 6, \dots\}$-Factor

Selecting Vertices → Generalized Domination

Fix two sets $\sigma, \rho \subseteq \mathbb{N}$.

(σ, ρ)-DomSet [Telle 1994]

Given: A simple graph G.

Is there a *vertex set* $S \subseteq V(G)$ such that
– for each $v \in V(G) \cap S$, we have $|N(v) \cap S| \in \sigma$, and
– for each $v \in V(G) \setminus S$, we have $|N(v) \cap S| \in \rho$?

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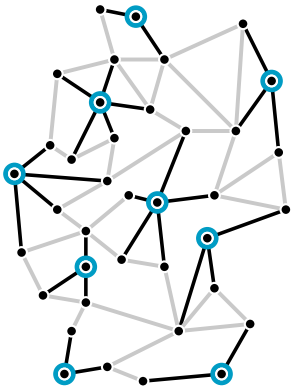

$(\{0\}, \{1\})$-DomSet

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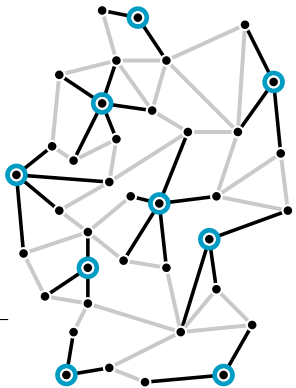
	Sel.	Uns.	σ	ρ
Total 2-Dom.	≥ 1	≥ 2	$\{1, 2, \dots\}$	$\{2, 3, \dots\}$
Independent Set	$= 0$	≥ 0	$\{0\}$	$\mathbb{N}$
Dominating Set	≥ 0	≥ 1	$\mathbb{N}$	$\{1, 2, \dots\}$
Perfect Code	$= 0$	$= 1$	$\{0\}$	$\{1\}$

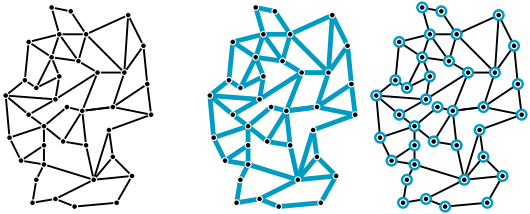


$(\{0\}, \{1\})$-DomSet

Solving the Problems

Few cases are trivial:

- $0 \in \Lambda$ or $0 \in \rho \rightarrow$ select nothing
- $\Lambda = \sigma = \mathbb{N} \rightarrow$ select everything



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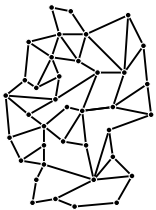

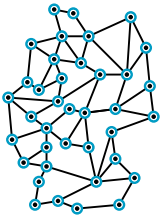

Our approach to the general cases:

- Restrict input graphs

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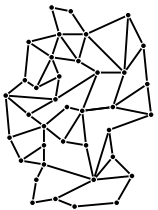

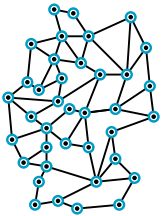

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- Study parameterization by *treewidth*

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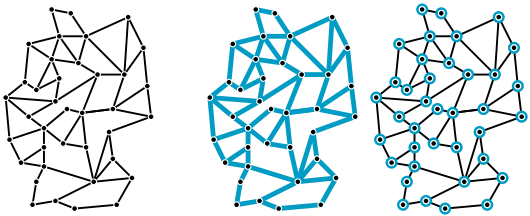


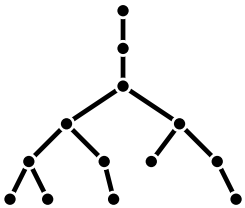
Our approach to the general cases:

- ~~Restrict input graphs~~
- Study parameterization by *treewidth*
- Consider *finite* and *cofinite* sets (i.e., the set or its complement is finite)
  $\rightsquigarrow$ covers most classical problems already

Interlude: Treewidth

Motivation

Frequently straight-forward algorithms for trees



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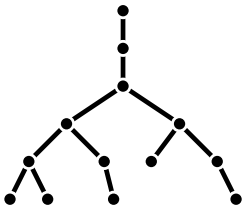

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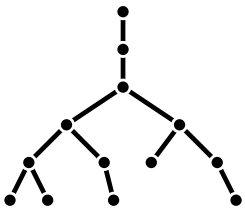

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- measures how “tree-like” a graph is.

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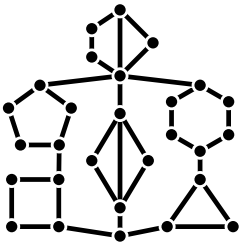

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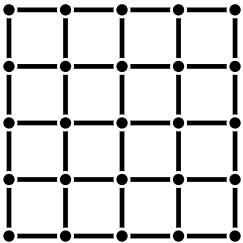


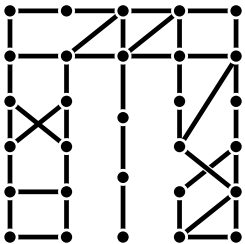
Large treewidth!

Interlude: Treewidth

Motivation

Frequently straight-forward algorithms for trees

$\rightsquigarrow$ Extend to general graphs



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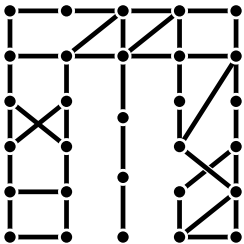

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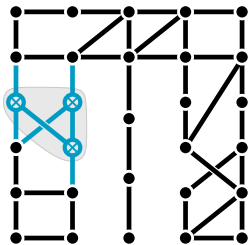

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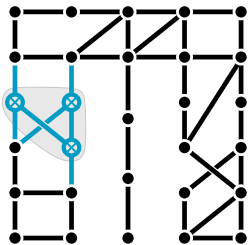

Treewidth …

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- has a definition that is hard to digest.
- is determined by special separators, referred to as “bags”.
- depends on some tree structure that allows for DPs (“tree decomposition”).

Solving the Problems: Known Results

Generalized Factor:

- Set Λ has no large “gap”: polynomial-time
  [Cornuéjols 1988]

Generalized Domination:

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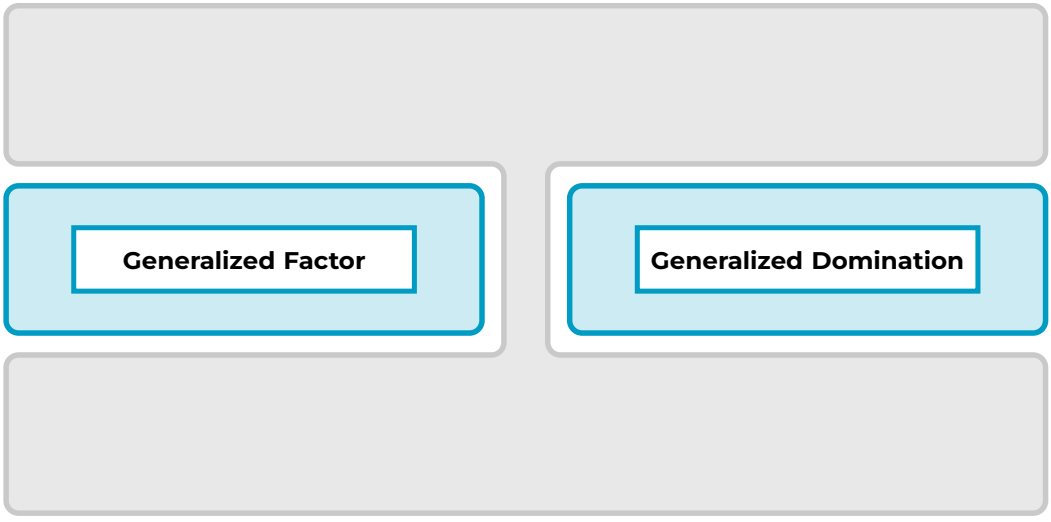
What about more general sets?

Are the known algorithms optimal?

What about counting the number of solutions?

Outline



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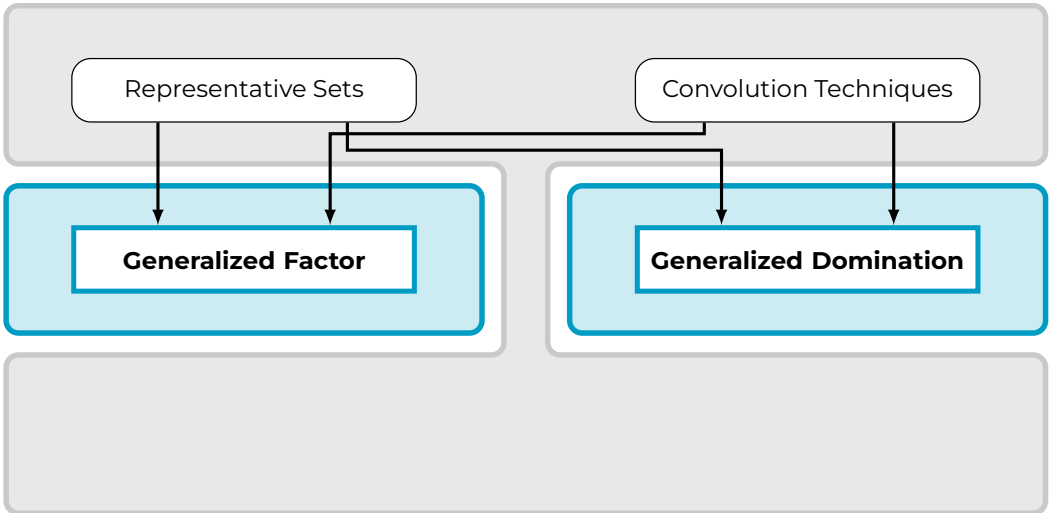
Representative Sets
Convolution Techniques
Generalized Factor
Generalized Domination

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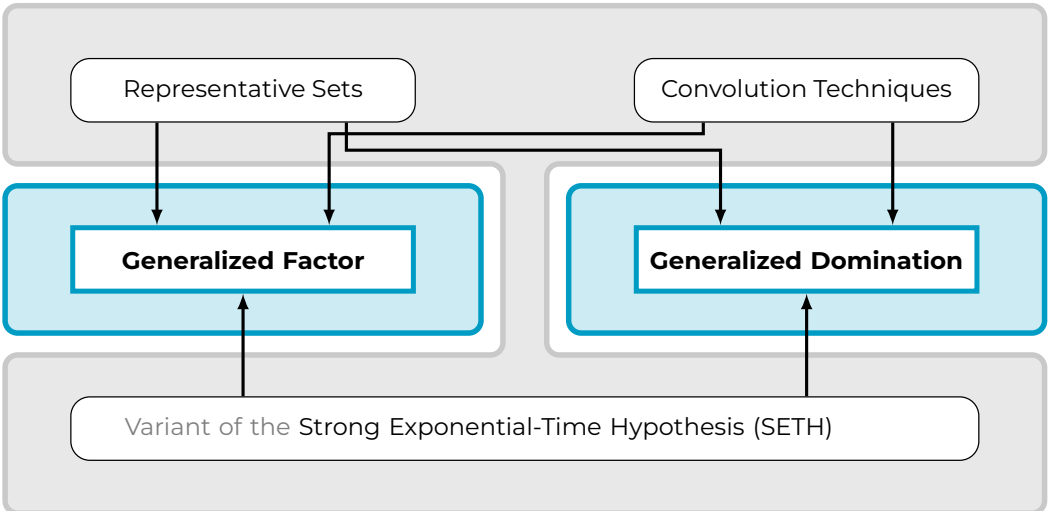
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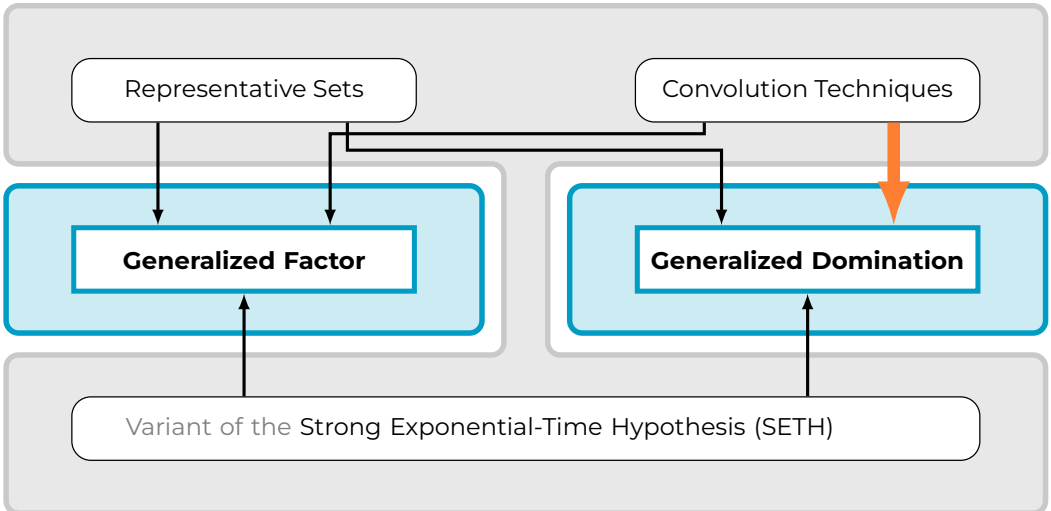
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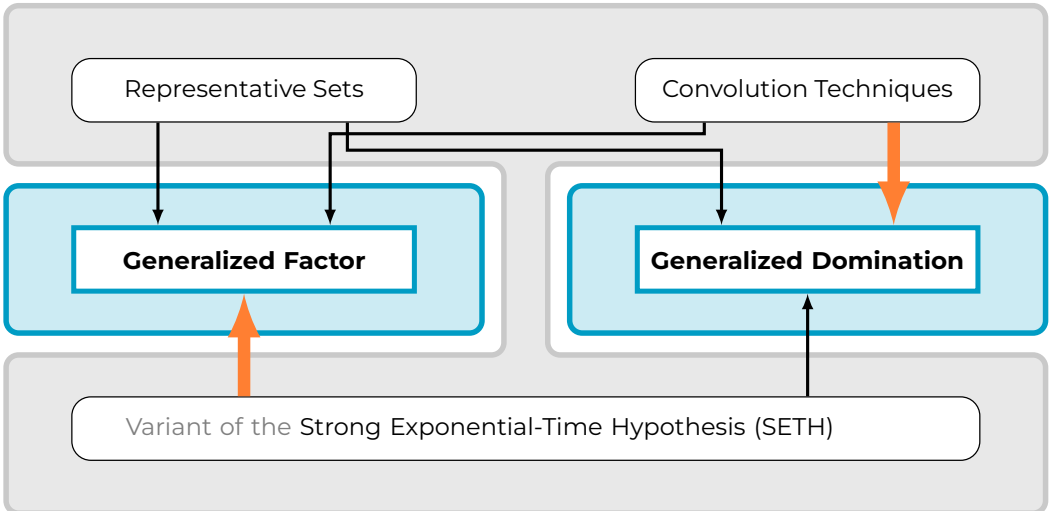
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Case Study: Perfect Code = (σ, ρ)-DomSet with $\sigma = \{0\}$ and $\rho = \{1\}$

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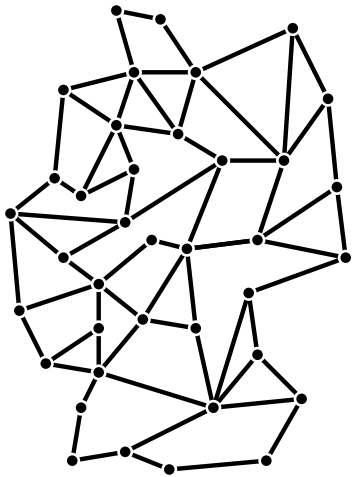

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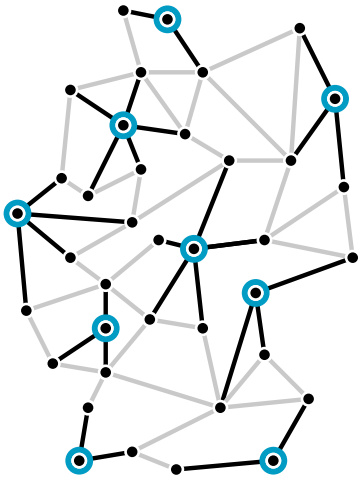

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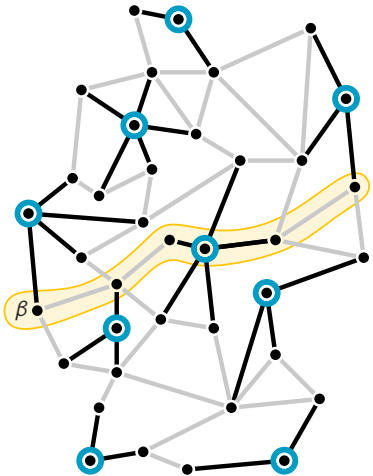

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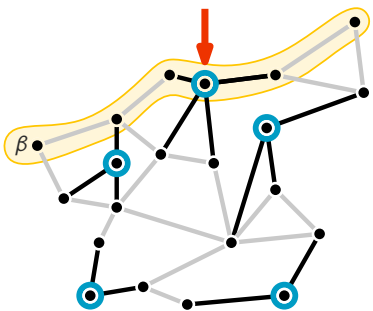

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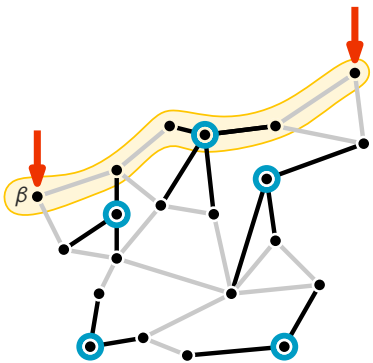

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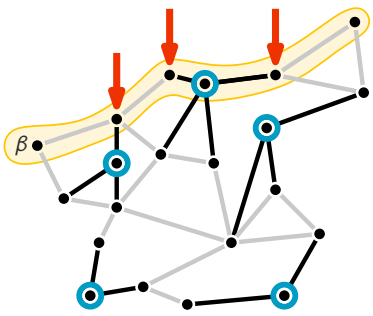

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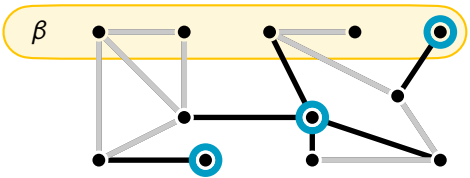
Can we do better? **Yes!**

Solving (σ, ρ)-DomSet: Structural Insights



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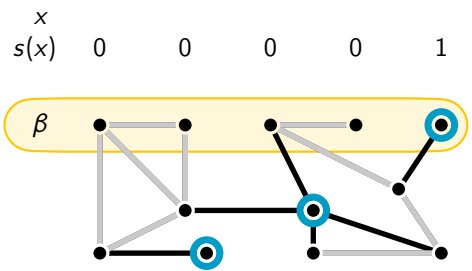

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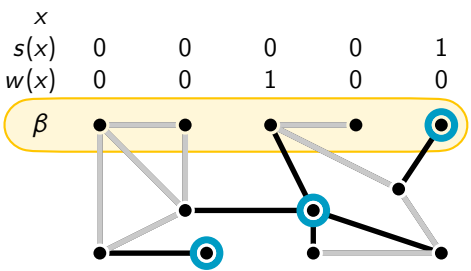

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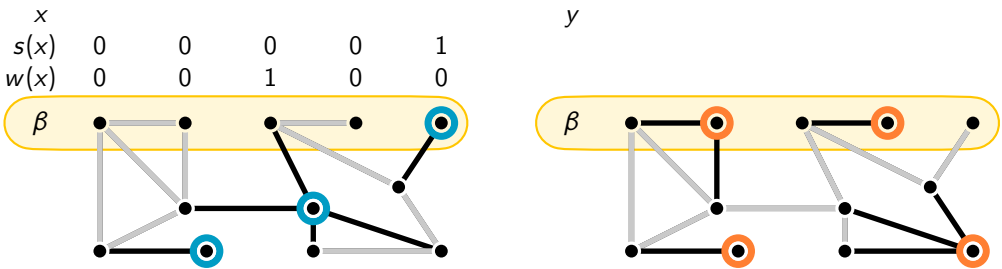

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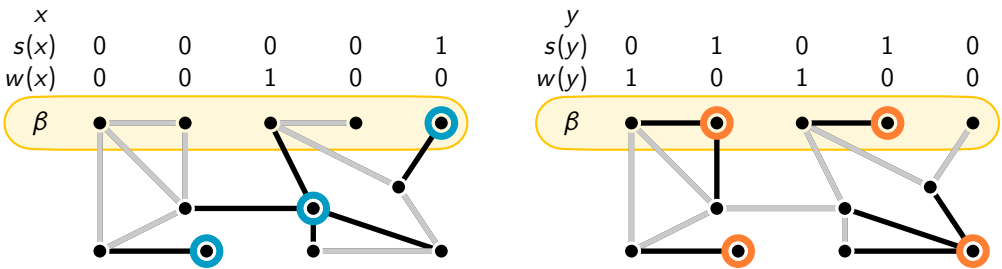

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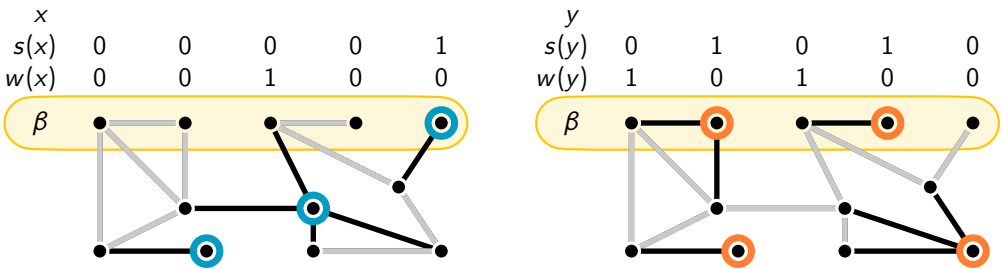

- Vectors are orthogonal: $s(x) \cdot w(y) = s(y) \cdot w(x) = 0$

$\rightsquigarrow$ Not too many vectors/solutions can satisfy this property

Solving (σ, ρ)-DomSet: A Faster Algorithm

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We extend this to other sets as well!

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For finite or cofinite $\sigma, \rho \subseteq \mathbb{N}$, we define a constant $c_{\sigma,\rho}$ such that

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Next goal: Check if constant $c_{\sigma,\rho}$ is optimal.

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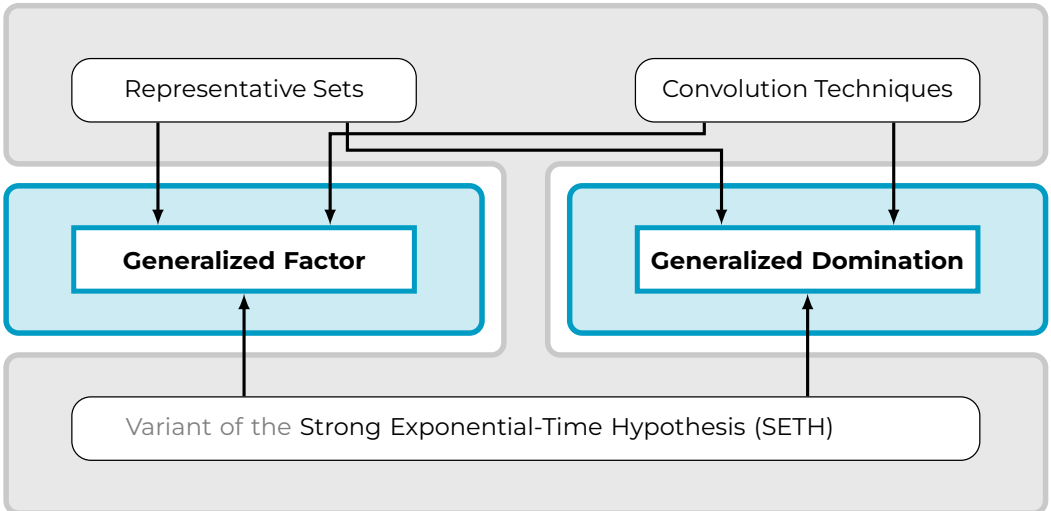
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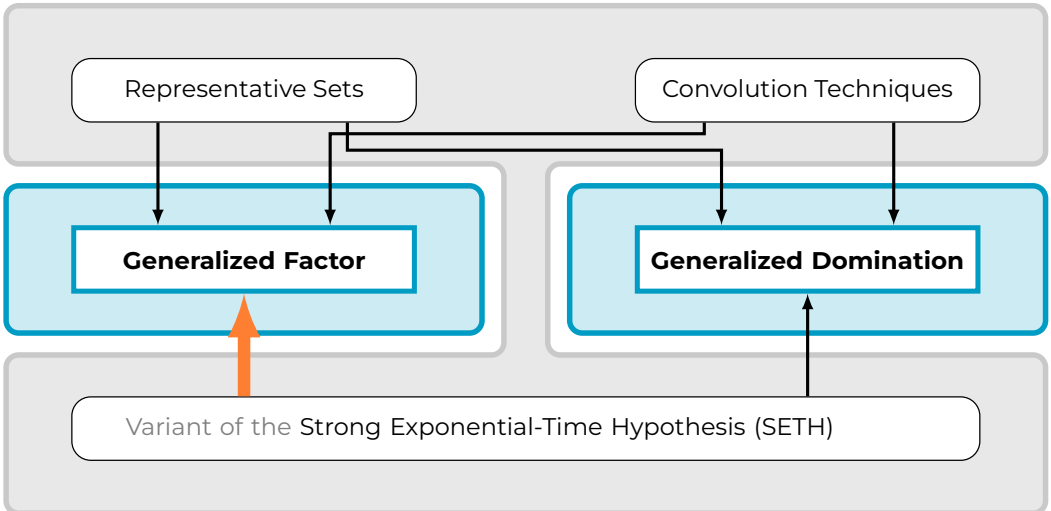
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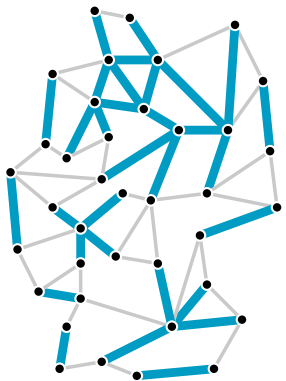
Λ-Factor: Upper Bound

For simplicity, consider finite sets for now.

Each vertex can have $\max \Lambda + 1$ intermediate states.



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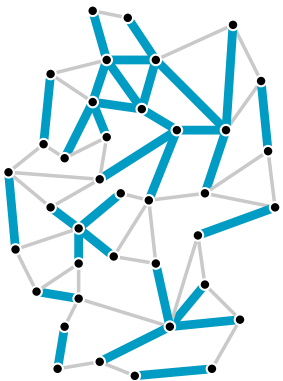

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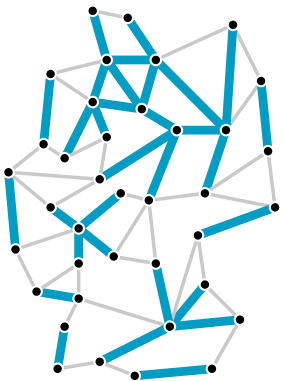
Prove:

No algorithm with running time $(\max \Lambda + 1 - \varepsilon)^{\mathrm{tw}} \cdot \mathrm{poly}(|G|)$ exists.



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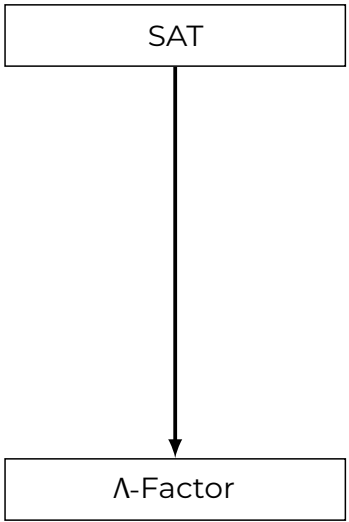

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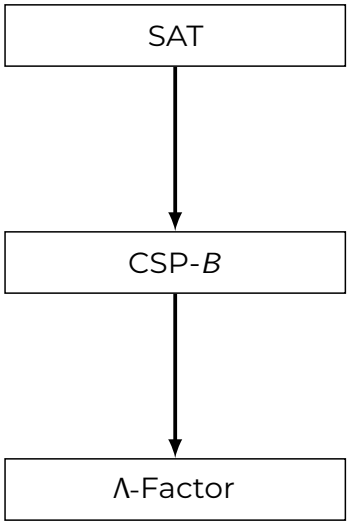
  [Lampis 2020 & 2025]



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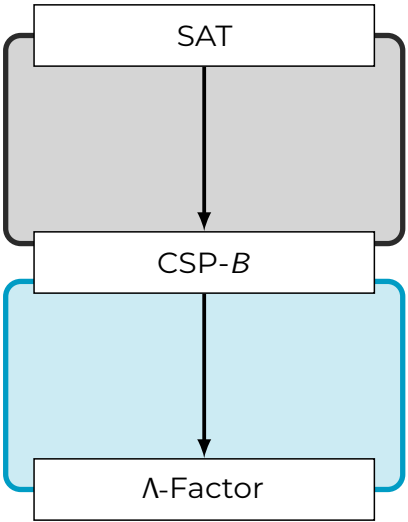
⇝ Simplified reductions that are easier to understand



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CSP-$(\max \Lambda + 1)$

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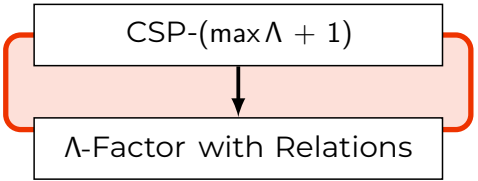

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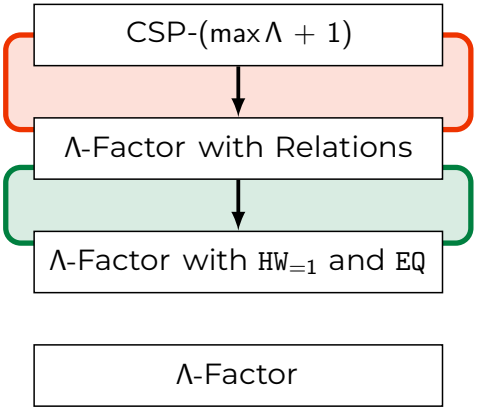

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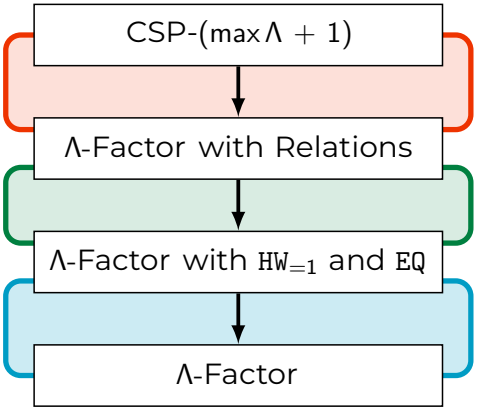

Encode the variable assignment
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Λ-Factor: Our Contribution

Fix a finite or cofinite set $\Lambda \subseteq \mathbb{N}$.

Define $\operatorname{top}\Lambda := \max\Lambda$ if Λ is finite and $\operatorname{top}\Lambda := \max(\mathbb{Z} \setminus \Lambda) + 1$ if Λ is cofinite.

Example: $\operatorname{top}\{1, 4\} = 4$ and $\operatorname{top}\{1, 4, 5, 6, \dots\} = 4$

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Decision version:

- **Finite set:** Similar lower bound
- **Cofinite set:** Improved algorithm but no matching lower bound

(σ, ρ)-DomSet: Our Contribution

Fix two finite or cofinite sets $\sigma, \rho \subseteq \mathbb{N}$.

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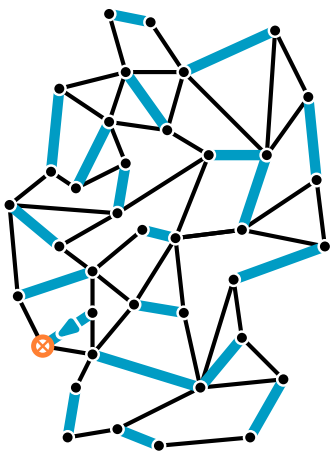
Open questions: Close gaps? More general sets? Other parameters?

Fire Protection Train (FPT) Problem

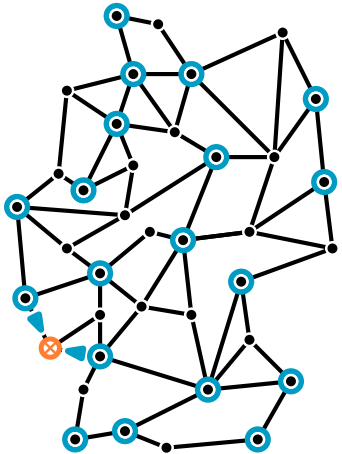
Task:

Find (few) locations to position the fire trains at.

Moving Train



Stationary Position



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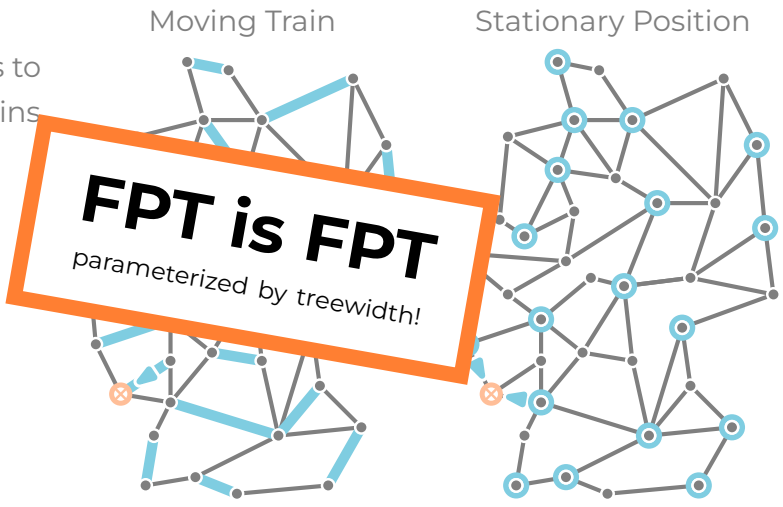
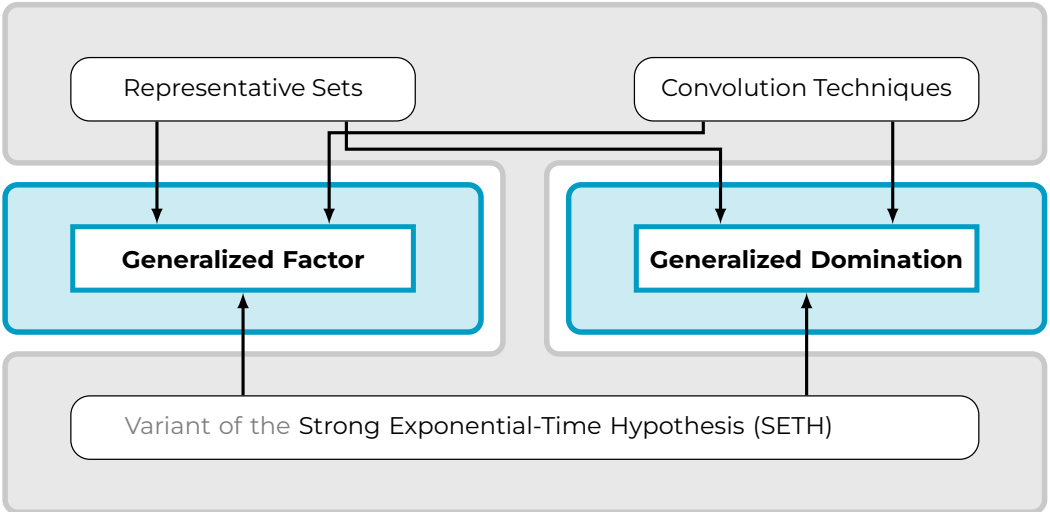


Appendix

Outline


Representative Sets
Convolution Techniques
Generalized Factor
Generalized Domination
Variant of the Strong Exponential-Time Hypothesis (SETH)

(σ, ρ)-DomSet: Structured Pairs

m-structured (σ, ρ)

For $m \geq 2$, (σ, ρ) is m-structured if there are α and β such that

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$\{0, 4\}$	$\{1, 9\}$	$m = 2, 4$
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If (σ, ρ) is m-structured for some $m \geq 2$, we show an improved algorithm.

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Definition for finite or cofinite $\sigma, \rho \subseteq \mathbb{N}$

- $c_{\sigma,\rho} \coloneqq \operatorname{top}\sigma + \operatorname{top}\rho + 2$ if (σ, ρ) is not $\mathtt{m}$-structured,
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Idea: Exploit this for a faster algorithm.

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We show a similar result for (σ, ρ)-DomSet.

Note: Incompatible with counting version (“every solution matters”)!

Half-Induced Matchings

Definition (Half-Induced Matching)

Graph $G = (L \cup R, E)$ has a half-induced matching of size ℓ if there are $a_1, \ldots, a_\ell \in L$ and $b_1, \ldots, b_\ell \in R$ such that

- $(a_i, b_i) \in E$ for all i, and
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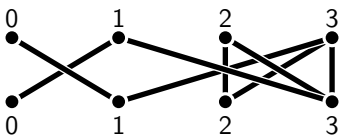

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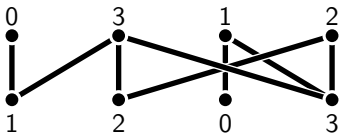

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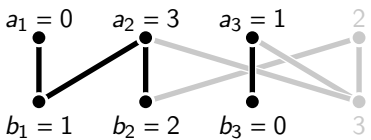

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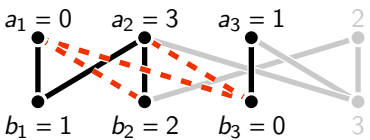
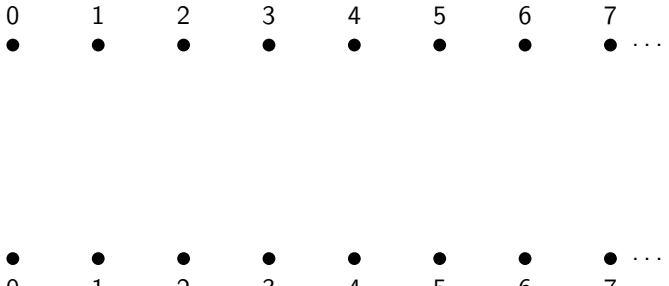


Half-induced matching of size 3

Half-Induced Matchings: Application

Consider Λ-Factor with $\Lambda = \mathbb{N} \setminus \{0, 3, 6\}$

Draw edge from i to j if $i + j \in \Lambda$:



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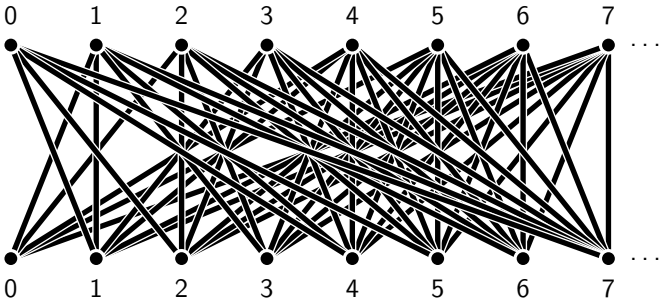

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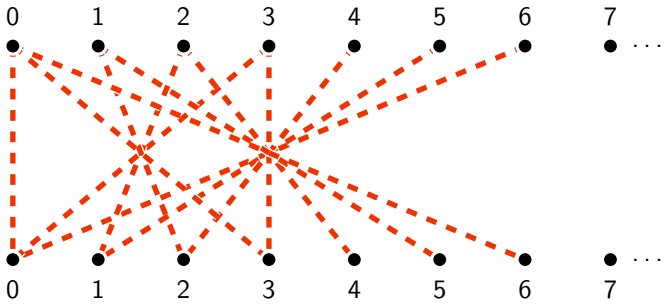

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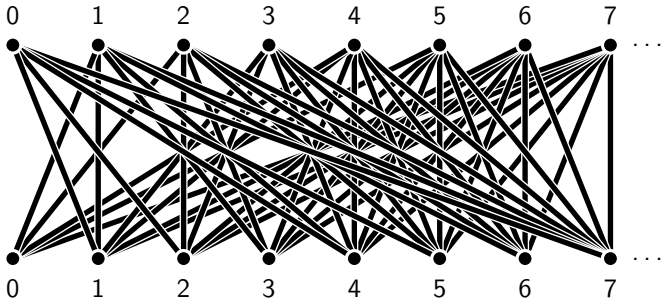

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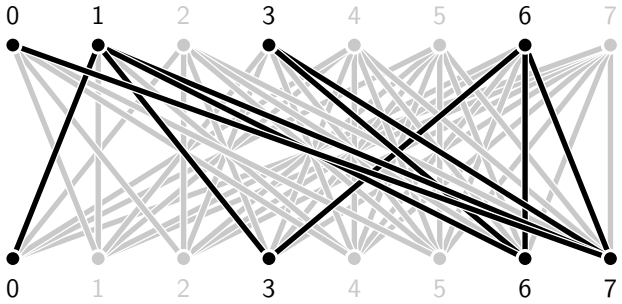

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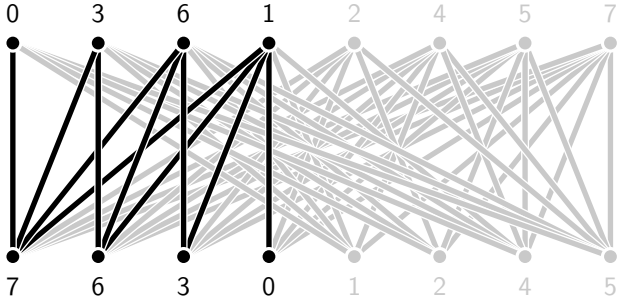

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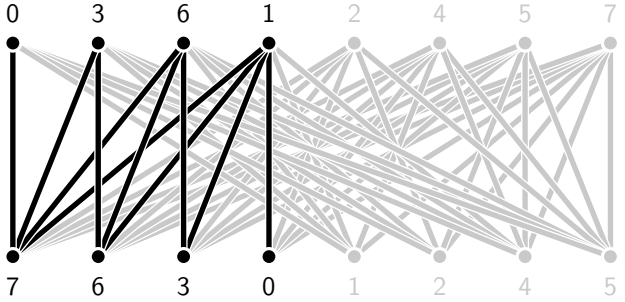


Our observation:

A larger half-induced matching provides a better lower bound!

Convolution Techniques

Extend algorithm by van Rooij in multiple steps: [van Rooij 2020]

- 𝕊 **Cyclic convolution:**
  Extend existing algorithm by computation of prime numbers
- 𝔸 **Normal addition:**
  Use cyclic convolution and encode checksum by 𝕃 to detect overflows
- 𝕃 **Normal addition with large numbers:**
  Use cyclic convolution and encode checksum in binary to detect overflows
- 𝕋 **Truncated addition:**
  Use normal addition and zeta- and Möbius-transform

Lower Bound: Known Hypotheses

Strong Exponential Time Hypothesis (SETH) [IP 2001, CIP 2009]

For all $\delta > 0$, there is a $k \geq 3$ such that satisfiability of k-CNF formulas on n variables cannot be solved in time $(2 - \delta)^n$.

Constraint Satisfaction Hypothesis (CSPH) [Lampis 2020]

For all $B \geq 2$ and $\varepsilon > 0$, there exists a q such that q-CSP-B with n variables cannot be solved in time $(B - \varepsilon)^n \cdot n^{\mathcal{O}(1)}$.

Primal Pathwidth SETH (pw-SETH) [Lampis 2025]

For all $\varepsilon > 0$, there is no algorithm which takes as input a 3-SAT instance ϕ and a path decomposition of its primal graph of width pw and correctly decides if ϕ is satisfiable in time $(2 - \varepsilon)^{\mathrm{pw}} \cdot |\phi|^{\mathcal{O}(1)}$.

Lower Bound: New Hypothesis

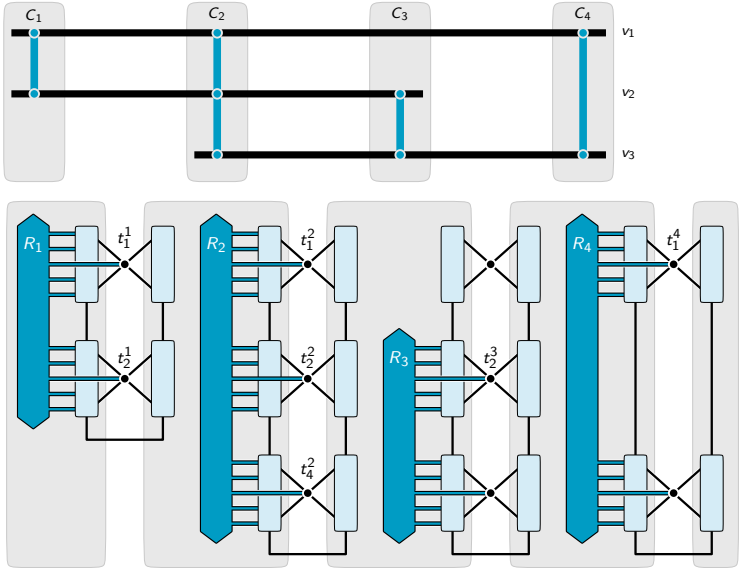
mono-pw-CSPH

For all $B \geq 2$ and $\varepsilon > 0$, there exists a q such that no algorithm can, for a q-CSP-B instance ϕ that is given with

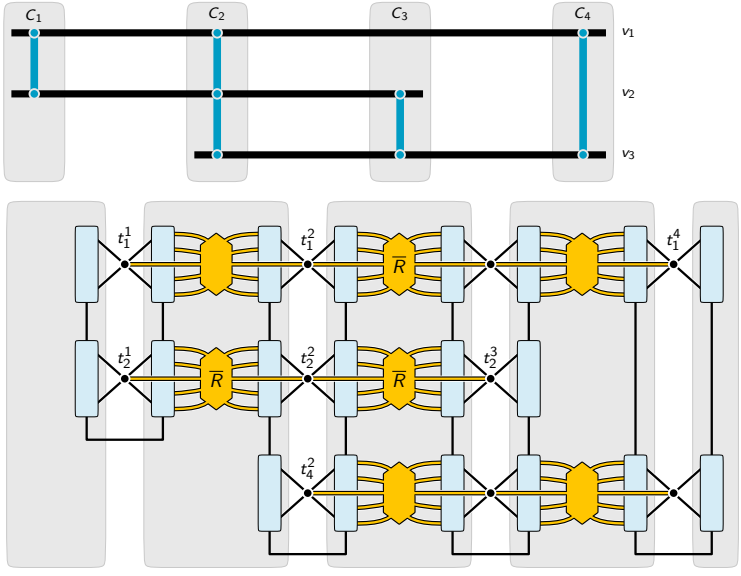
- a partition $V_1 \dot{\cup} V_2$ of the variables,
- a nice path decomposition of the primal graph of ϕ of width w
- where every bag contains at most $\mathcal{O}(B \cdot \log|V_1|)$ variables from V_2,
- an injective mapping c from the constraints of ϕ to the bags,
- the promise that every satisfying multi-assignment that is consistent on V_2 is also consistent on V_1,

decide if there is a monotone satisfying multi-assignment that is consistent on V_2 in time $(B-\varepsilon)^w \cdot |\phi|^{\mathcal{O}(1)}$.

Lower Bound: High-Level Construction



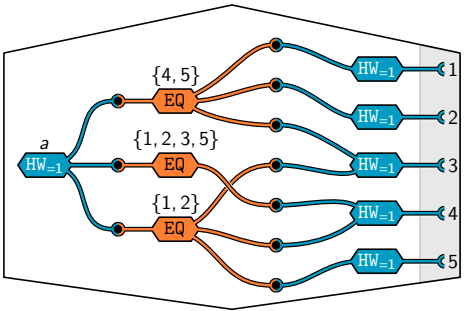
Lower Bound: High-Level Construction



Lower Bound: Realizing Relations

To realize arbitrary relations it suffices to

- force a single vertex/edge (of a group) to be selected $\rightsquigarrow \texttt{HW}_{=1}$ and
- ensure that a group of vertices/edges is consistently selected (all or none) $\rightsquigarrow \texttt{EQ}$.

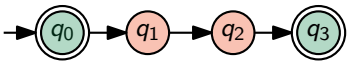


Summary (Formalized)

Problem	Algorithm	Lower Bound	
(σ, ρ)-#DomSet	$(c_{\sigma,\rho})^{\mathrm{tw}}$	no $(c_{\sigma,\rho} - \varepsilon)^{\mathrm{pw}}$	
(σ, ρ)-DomSet	$(c_{\sigma,\rho})^{\mathrm{tw}}$	no $(c_{\sigma,\rho} - \varepsilon)^{\mathrm{pw}}$	σ and ρ finite
	$(\mathrm{cost}_{\sigma,\rho})^{\mathcal{O}(\mathrm{tw})}$	open	σ or ρ cofinite
#AntiFactor$_x$	n^{tw}	no $n^{\mathrm{pw}-\varepsilon}$	
Λ-#Factor	$(\mathrm{top}\,\Lambda + 1)^{\mathrm{tw}}$	no $(\mathrm{top}\,\Lambda + 1 - \varepsilon)^{\mathrm{pw}}$	
Λ-Factor		no $(\mathrm{top}\,\Lambda + 1 - \varepsilon)^{\mathrm{pw}}$	Λ finite
		no $(h - \varepsilon)^{\mathrm{pw}}$	Λ cofinite with half-induced matching of size h
AntiFactor$_x$	$(x+1)^{\mathcal{O}(\mathrm{tw})}$	no $(x + 1 - \varepsilon)^{\mathrm{pw}}$	x degrees forbidden

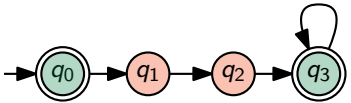
More General Sets: From States to Automata

Finite set:



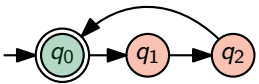
$\tau = \{0, 3\}$

Cofinite set:



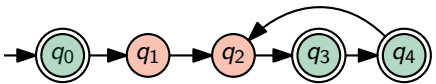
$\tau = \{0, 3, 4, 5, 6, \dots\}$

Periodic set (residue class)**:**



$\tau = \{0, 3, 6, 9, \dots\}$

Infinite set (ultimately periodic)**:**



$\tau = \{0, 3, 4, 6, 7, 9, 10, \dots\}$

Next steps:

Connection to *faster* algorithms? Efficient convolution techniques?