CISPA HELMHOLTZ CENTER FOR INFORMATION SECURITY

Computing Generalized Convolutions Faster Than Brute Force

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Motivation

Convolutions are used in parameterized and exact algorithms for

- Hamiltonian Cycle,
- Feedback Vertex Set,
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Normally each problem needs a separate type of convolution.

Goal:
Unify the convolution procedures under one umbrella.

Convolutions

Classical convolution:
For two functions $g, h\colon \mathbb{Z} \to \mathbb{Z}$ this is

$$(g * h)(c) \coloneqq \sum_{a} g(a) \cdot h(c - a) = \sum_{a+b=c} g(a) \cdot h(b) \qquad \forall\, c \in \mathbb{Z}.$$

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We *do not* study this!

Cover Product:

Let S be a set. For two functions $g, h\colon 2^S \to \mathbb{Z}$, the cover product is

$$(g *_{\mathrm{CP}} h)(C) \coloneqq \sum_{A \cup B = C} g(A) \cdot h(B) \qquad \forall\, C \subseteq S.$$

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$$(g \circledast_f h)(\mathbf{v}) \coloneqq \sum_{\substack{\mathbf{u},\mathbf{w} \in D^n \\ \text{s.t. } \mathbf{v} = f(\mathbf{u},\mathbf{w})}} g(\mathbf{u}) \cdot h(\mathbf{w}) \qquad \forall\, \mathbf{v} \in D^n.$$

Here, $f(\mathbf{u}, \mathbf{v})$ denotes the coordinate-wise application of f for two vectors $\mathbf{u}, \mathbf{v} \in D^n$.

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Here, $f(\mathbf{u}, \mathbf{v})$ denotes the coordinate-wise application of f for two vectors $\mathbf{u}, \mathbf{v} \in D^n$.
For Cover Product we set $D = \{0, 1\}$ and f as addition with maximum of 1.

Computing Generalized Convolutions

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Theorem (Brute Force Approach)

f-CONVOLUTION can be solved in time $|D|^{2n} \cdot n^{\mathcal{O}(1)}$ by a brute-force approach.

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Only for special cases faster algorithms are known.

Theorem (van Rooij 2021, Umans 2019 + Yates 1937)

f-CONVOLUTION can be solved

- in time $|D|^{n} \cdot n^{\mathcal{O}(1)}$ if f is addition (with maximum or modulo), or maximum, and
- in time $|D|^{\omega \cdot n/2} \cdot n^{\mathcal{O}(1)}$ if f is a finite-group operation

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We improve the naive computation for *all* functions f.

Main Theorem (simplified)

f-CONVOLUTION can be solved in time $(\frac{5}{6}|D|^2)^n \cdot n^{\mathcal{O}(1)}$ for all $f\colon D \times D \to D$.

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Example for $|D| = 6$:

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1. Find a small cyclic partition of the function f.

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Main idea: "Reduce" f-Convolution to addition with modulo.

1. Find a small cyclic partition of the function f.
2. Give a general algorithm based on cyclic partitions to compute the convolution.

Cyclic Minors

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Definition (Cyclic Minor)

For $A, B \subseteq D$ and $k \in \mathbb{N}$, (A, B, k) is a *cyclic minor* of f if the restriction of f to A and B is addition modulo k, after relabeling the sets A, B, and D.

f	a	b	c	d
a	a	d	b	d
b	c	a	d	b
c	b	b	a	c
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The function $f : D \times D \to D$ with $D = \{a, b, c, d\}$.

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The function f restricted to $A = \{b, c, d\}$ and $B = \{b, d\}$.

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$+$		0		1
0		0		1
1		1		2
2		2		0

A relabeling shows that (A, B, k) is a cyclic minor of f with $A = \{b, c, d\}$, $B = \{b, d\}$, and $k = 3$.

Cyclic Partitions

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A *cyclic partition* of f is a set $\mathcal{P} = \{(A_1, B_1, k_1), \ldots, (A_m, B_m, k_m)\}$ if (A_i, B_i, k_i) is a cyclic minor of f and $A_1 \times B_1, \ldots, A_m \times B_m$ is a partition of $D \times D$.
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A cyclic partition of f with cost 8:

$$\mathcal{P} = \{(\{b, c, d\}, \{b, d\}, 3), (\{a, c\}, \{a, c\}, 2), (\{b, d\}, \{a, c\}, 2), (\{a\}, \{b, d\}, 1)\}.$$

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Lemma (Cyclic Partitions are Useful)

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There is a $(\text{cost}(\mathcal{P})^n + |D|^n) \cdot n^{\mathcal{O}(1)}$ time algorithm for f-CONVOLUTION.

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This improves $\mathbf{4^{2n}} \cdot n^{\mathcal{O}(1)} = \mathbf{16^n} \cdot n^{\mathcal{O}(1)}$ to $(\mathbf{8^n + 4^n}) \cdot n^{\mathcal{O}(1)} = \mathbf{8^n} \cdot n^{\mathcal{O}(1)}$.

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We show how to find a "good" cyclic partition.

Finding Cyclic Partitions: Part I

Lemma (Existence of Non-Trivial Cyclic Partitions)

There is a cyclic partition $\mathcal{P}$ of f such that $\text{cost}(\mathcal{P}) \leq \frac{5}{6} \cdot |D|^2$.

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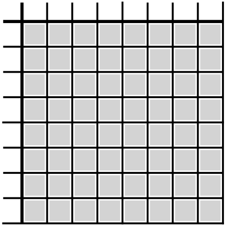

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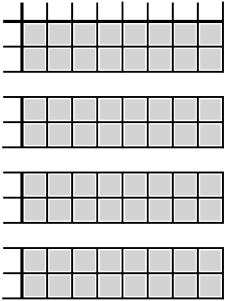

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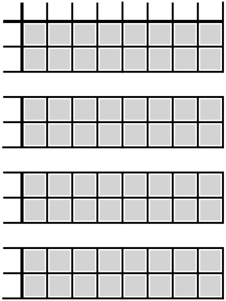

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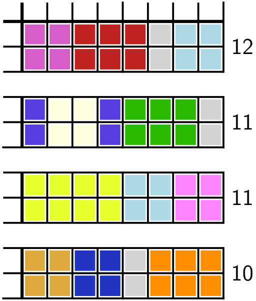

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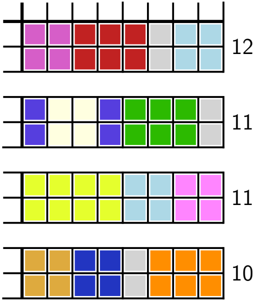

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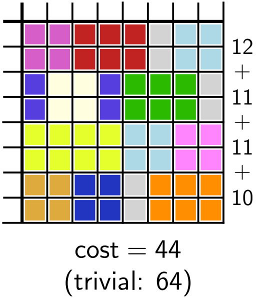

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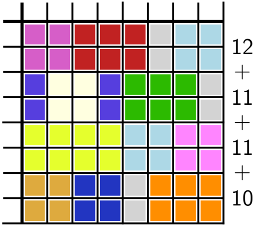
The cyclic partition $\mathcal{P}$ of f has cost

$$\text{cost}(\mathcal{P}) = \sum_{i=1}^{|D|/2} \text{cost}(\mathcal{P}_i) \leq \frac{|D|}{2} \cdot \frac{5}{3}|D| = \frac{5}{6}|D|^2.$$



cost = 44
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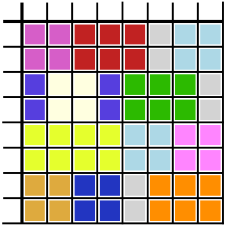

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Next: Find a "small" cyclic partition of f restricted to D_i and D.

Finding Cyclic Partitions: Part II

Consider f restricted to $D_i = \{x, y\}$ and D (assume $D = \{0, \ldots, 9\}$):

	0	1	2	3	4	5	6	7	8	9
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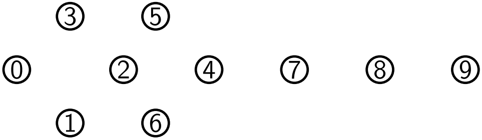
Construct a graph G with $V(G) = D$ and $E(G) = \{(f(x, d), f(y, d)) \mid d \in D\}$.



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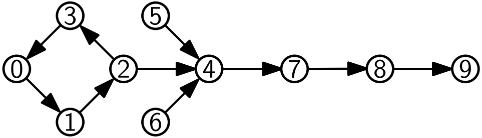

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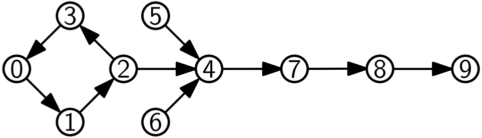

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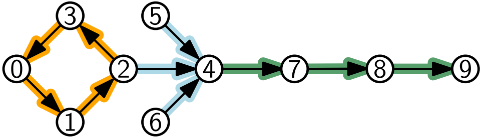

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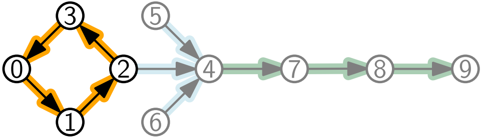

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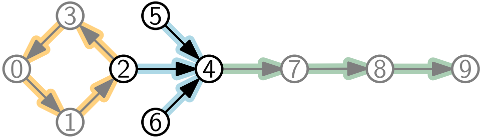

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y	1	2	3	0	4	4	4	7	8	9	4

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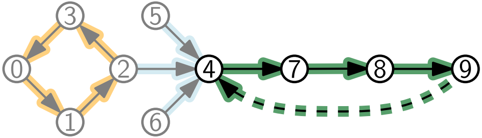

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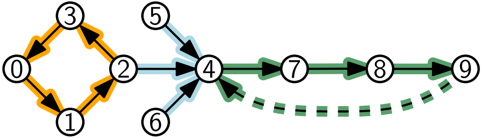

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Our bound is obtained by balancing different ways of decomposing the graph.

Conclusion

Main Theorem

Let D be a finite set and $f : D \times D \to D$. There is an algorithm for f-Convolution with running time $(\frac{5}{6} \cdot |D|^2)^n \cdot n^{\mathcal{O}(1)}$ when $|D|$ is even, or $(\frac{5}{6} \cdot |D|^2 + \frac{1}{6} \cdot |D|)^n \cdot n^{\mathcal{O}(1)}$ when $|D|$ is odd.

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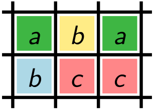
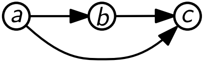
Full version: `arXiv:2209.01623`

Special Structures

Improvements for the following cases might give helpful insights:

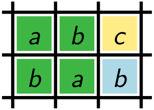


Graph representation:

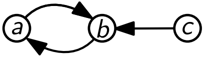


Current cost: 4

Expected cost: 3

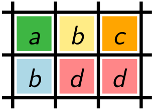


Graph representation:

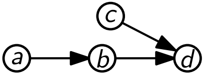


Current cost: 4

Expected cost: 4



Graph representation:



Current cost: 5

Expected cost: 4