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Anti-Factor is FPT Parameterized by Treewidth and List Size (but Counting is Hard)

IPEC 2022

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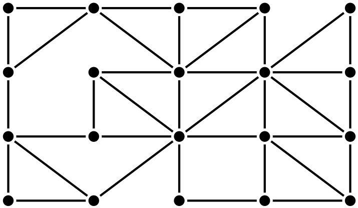
September 8, 2022

The Classical Matching Problem

Definition (Perfect Matching)

Input: A simple graph $G = (V, E)$.
Task: Check if there is a set $S \subseteq E$ such that $\deg_S(v) = 1$, for all $v \in V$.



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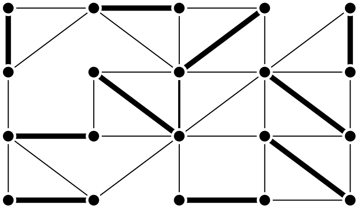

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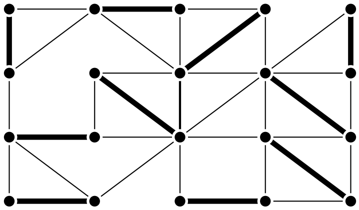

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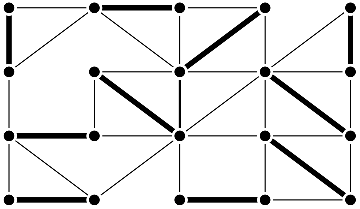

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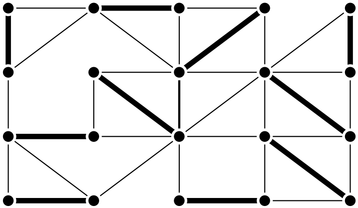

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Lovász introduced 1972 a general version called B-FACTOR.

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Fixed: A finite, non-empty set $B \subseteq \mathbb{N}$.
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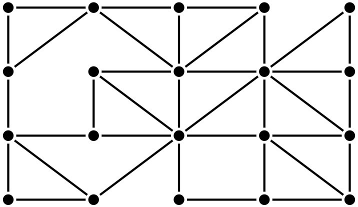

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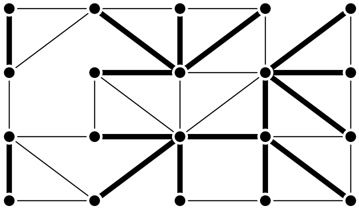

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Complexity of B-Factor is understood.

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Until now we *allowed* degrees. What changes if we *exclude* degrees?

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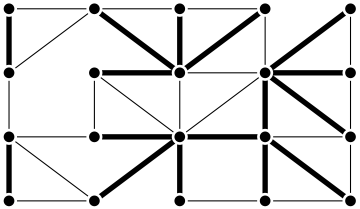

Solution for B-Factor

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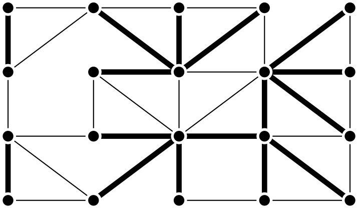

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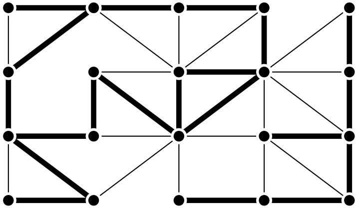

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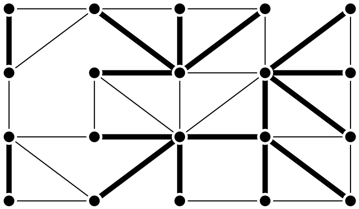

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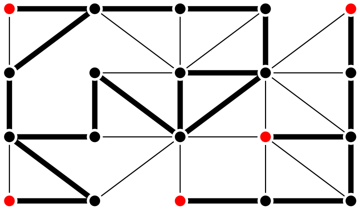

Complement of the solution for B-Factor does not work.

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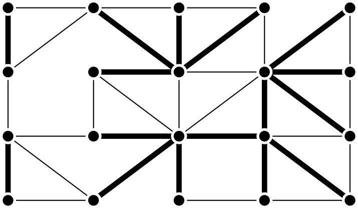

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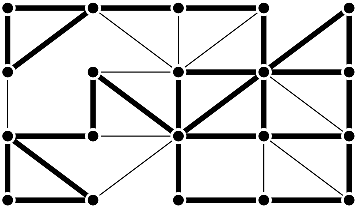

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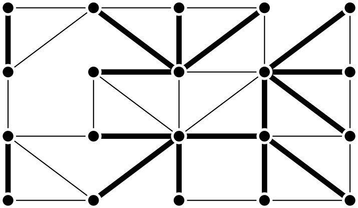

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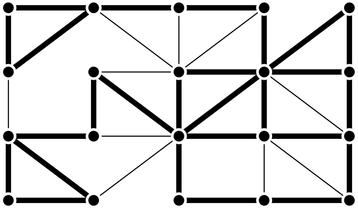

A correct solution for X-ANTIFACTOR

Which running time is needed to solve X-ANTIFACTOR?

Solving X-ANTIFACTOR

We extend the algorithm for B-FACTOR to solve X-ANTIFACTOR.

Theorem (Parameterize by $\max X$)

Let $X \subseteq \mathbb{N}$ be finite and fixed. X-ANTIFACTOR can be solved in time $(\max X + 2)^{\mathrm{tw}} n^{\mathcal{O}(1)}$ assuming a tree decomposition of width tw is given.

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Is this algorithm optimal?

For example:
Is the $1002^{\text{tw}} n^{\mathcal{O}(1)}$ algorithm for $\{0, 999, 1000\}$-ANTIFACTOR optimal?

Excluding Only Few Numbers

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Theorem (Parameterize by $|X|$)

Let $X \subseteq \mathbb{N}$ be finite and fixed. X-AntiFactor can be solved in time $(|X|+1)^{4\text{tw}} n^{\mathcal{O}(1)}$ assuming a tree decomposition of width tw is given.

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Consider $X = \{0, 999, 1000\}$.

Running time:
Improves from $1002^{\mathrm{tw}} n^{\mathcal{O}(1)}$ to $4^{4\mathrm{tw}} n^{\mathcal{O}(1)} = 256^{\mathrm{tw}} n^{\mathcal{O}(1)}$.

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Intuition:

- Fix a vertex v.

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Intuition:

- Fix a vertex v.
- Fix four solutions which give degree 0, 300, 500, and 999 to v, respectively.

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- Fix a vertex v.
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- A future extension increases the degree of V by f.

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- One of these four solutions can be combined with the new solution as $\{0+f, 300+f, 500+f, 999+f\} \neq \{0, 999, 1000\} = X$.

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Once more: Is this algorithm optimal (up to constants in the exponent)?

Lower Bounds for B-Factor [Marx et al. '21]

Reduction from SAT with n variables and m clauses to B-Factor:

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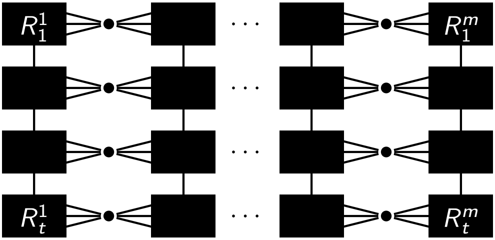

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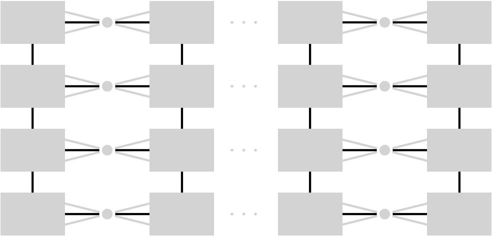

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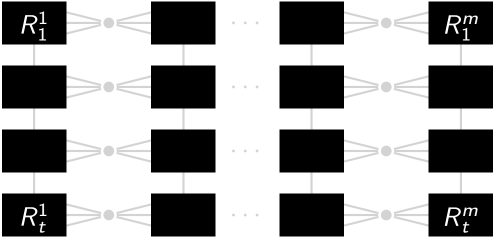

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1. Grid-like structure:
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2. Relations:
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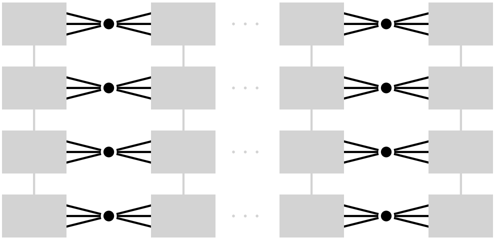


Main parts of the construction:

1 Grid-like structure:
Gives us a low treewidth

2 Relations:
Check if the clauses are satisfied

3 Information vertices:
Transfer information/assignment between columns/clauses

Lower Bounds for B-Factor [Marx et al. '21]

Reduction from SAT with n variables and m clauses to B-Factor:

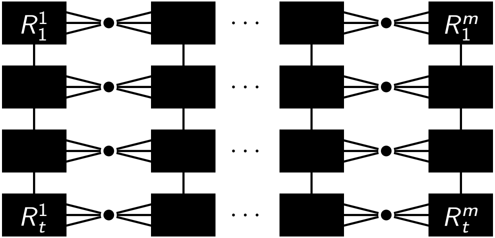
0 Construct a grid-like B-Factor instance.

1 Group $\approx \log(\max B)$ variables together.
Encode the assignments by the number of selected incident edges of a vertex.

2 Have one row for each group of variables.

3 Have one column for each clause of the formula.

4 At the crossing points check if the assignment satisfies the clause.



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Gives us a low treewidth

2 Relations:
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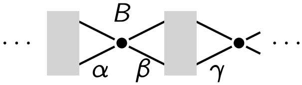
Transfer information/assignment between columns/clauses

Parts 1 and 2: They can be rather easily adopted for X-AntiFactor.
Part 3: Behavior of information vertices changes dramatically.

Understanding Information Vertices



- The number of selected edges to the left, i.e., α encodes the assignment.

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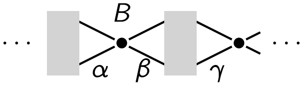

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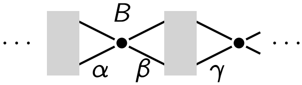

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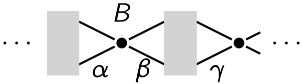

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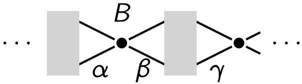

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$B = \{0, 2\}$:

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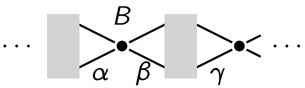

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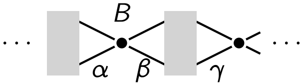

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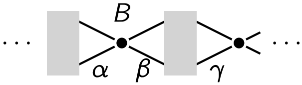

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Understanding Information Vertices

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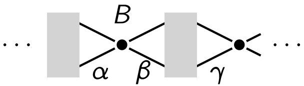

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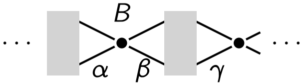

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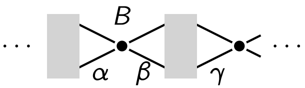

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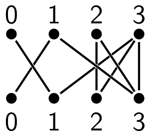
Have to use some other property of B, respective X!

Half-Induced Matchings

Fix a bipartite graph $G = (U \dot\cup V, E)$.

Definition (Induced Matching)

G has an induced matching of size ℓ, if there are two sets $A \subseteq U$ and $B \subseteq V$ of size ℓ such that the edges between A and B are a perfect matching of size ℓ.



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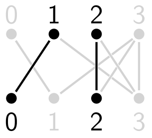

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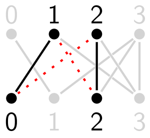

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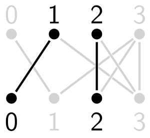

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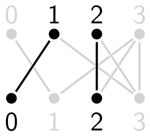

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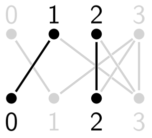

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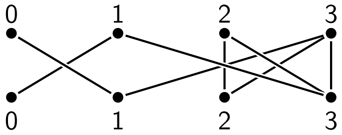

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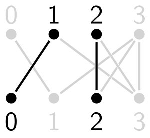

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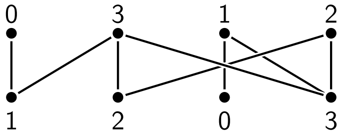

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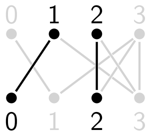

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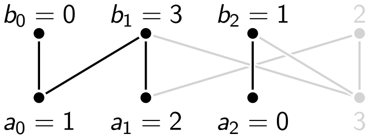

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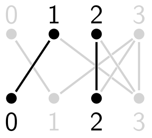

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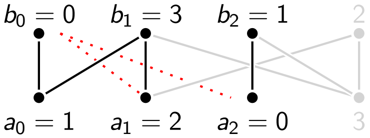
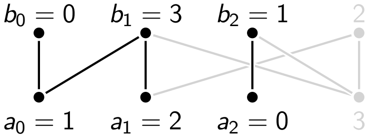


Fixing the Behavior of Information Vertices

We use half-induced matchings to modify the information vertices.



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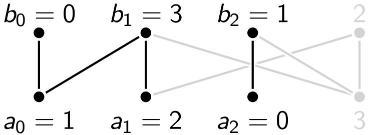

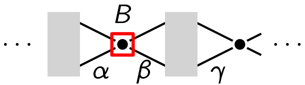

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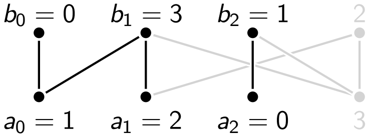

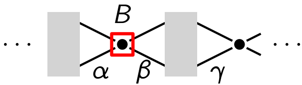

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$B = \{2\}$:

$\alpha \mapsto \beta \mapsto \gamma$

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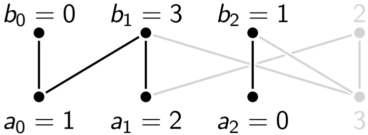

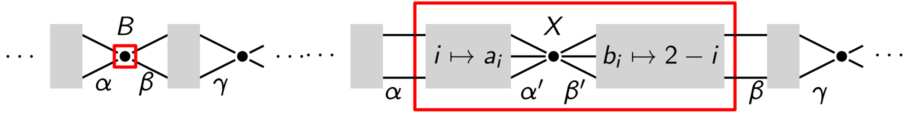

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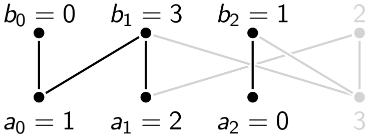

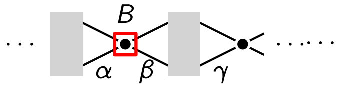

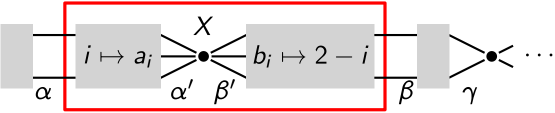

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α	$\mapsto \beta$	$\mapsto \gamma$
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$X = \{0, 2, 3\}$ ($B = \{1, 4, 5, \ldots\}$):

α	$\mapsto \alpha'$	$\mapsto \beta'$	$\mapsto \beta$	$\mapsto \gamma$
0	$\mapsto 1$	$\mapsto 0, 3$	$\mapsto 2, 1$	$\mapsto 0, 1$
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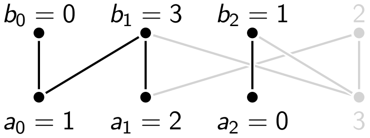

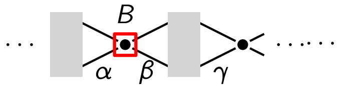

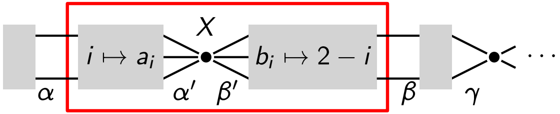

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$2 \mapsto$	$0 \mapsto$	2

After the modification.

$X = \{0, 2, 3\}$ ($B = \{1, 4, 5, \dots\}$):

$\alpha \mapsto$	β	$\mapsto \gamma$
$0 \mapsto$	$2, 1$	$\mapsto 0, 1$
$1 \mapsto$	1	$\mapsto 1$
$2 \mapsto$	0	$\mapsto 0$

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Theorem (Lower Bound)

Fix a finite set $X \subseteq \mathbb{N}$ such that X contains a half-induced matching of size h and $0 \in X$ and $\text{max-gap}(X) > 1$.
For any $\varepsilon > 0$, there is no $(h - \varepsilon)^{\text{tw}} n^{\mathcal{O}(1)}$ time algorithm for X-AntiFactor even if we are given a tree decomposition of width tw, unless SETH fails.

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- We conjecture that the half-induced matchings are the actual base for the lower bound of the problem.

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- Result holds already for the case when only one number is excluded.

Conclusion

Fix some finite, non-empty set $X \subseteq \mathbb{N}$.
Let h be the size of the largest half-induced matching in X.

Parameter	Decision Version	Counting Version
maximum		
forbidden degree		
list size*		

Always assuming

- the problem is not polynomial-time solvable (or trivial) because of X,
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Full version: `arXiv:2110.09369`

What Makes B-Factor Easy

Definition (Max-Gap)

For finite $\emptyset \neq B \subseteq \mathbb{N}$, max-gap B is the largest d such that at most d consecutive numbers are missing in B.
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What do we know about the parameterized version, e.g. for treewidth?