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Fine-Grained Complexity of Regular Expression Pattern Matching and Membership

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Motivation

Regular Expressions are used for

- Text analysis and manipulation (e.g. unix tools `grep` and `sed`)
- Network analysis
- Searching for proteins in DNA sequences
- Human-computer interaction

Definition (Membership)

Input: Text t of length n and pattern p of size m.
Question: Does p generate t, i.e. $t \in \mathcal{L}(p)$?

Definition (Pattern Matching)

Question: Does p generate some *substring* of t,
i.e. $t \in \mathcal{M}(p) := \Sigma^*\mathcal{L}(p)\Sigma^*$?

How fast can these problems be solved? $\mathcal{O}(nm)$! $o(nm)$? $\Omega(nm)$?

Agenda

1. Introduction and Results
2. Upper Bounds
3. Lower Bounds
4. Conclusion

Recap: Regular Expressions

Name	Regular Expression	Language
Symbol	σ	$\mathcal{L}(\sigma) := \{\sigma\}$
Alternative	$(p \mid q)$	$\mathcal{L}(p \mid q) := \mathcal{L}(p) \cup \mathcal{L}(q)$
Concatenation	$p \circ q$	$\mathcal{L}(p \circ q) := \{tu \mid t \in \mathcal{L}(p) \wedge u \in \mathcal{L}(q)\}$
Kleene Plus	p^+	$\mathcal{L}(p^+) := \bigcup_{i=1}^{\infty} \mathcal{L}(\underbrace{p \circ \ldots \circ p}_{i})$
Kleene Star	p^*	$\mathcal{L}(p^*) := \{\varepsilon\} \cup \mathcal{L}(p^+)$

n text length, m pattern size (=number of operators and symbols)

Current Results

Upper bounds

- Classical (Thompson 1968): $\mathcal{O}(nm)$
- Myers 1992: $\mathcal{O}(nm/\log n)$
- Bille and Thorup 2009: $\bar{\mathcal{O}}(nm/\log^{3/2} n)$ ($\bar{\mathcal{O}}$ hides $\operatorname{poly}\log\log n$ factors)

Lower bounds

- Backurs and Indyk 2016: $\Omega((nm)^{1-\epsilon})$,
  assuming the Strong Exponential Time Hypothesis (SETH)
- Abboud and Bringmann 2018: $\Omega(nm/\log^{7+\epsilon} n)$,
  assuming the Formula-Sat Hypothesis (FSH)

$\rightarrow$ Matching lower and upper bound up to a constant number of log-factors for the **general** case!

What about "**easier**" patterns? What is an "easy" pattern?

Homogeneous Patterns

Represent the patterns as trees:

- Leaves are labeled with symbols from Σ
- Inner nodes are labeled with operations ($|, \circ, +, \star$)

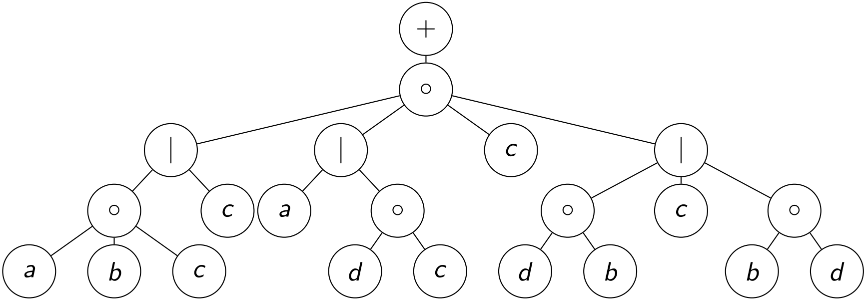
Definition (Homogeneous Patterns)

For each level of the corresponding tree the inner nodes have to be labeled with the same operation. The *type* is the sequence of operators on the path from the root to the deepest leaf.

Homogeneous Patterns – Example I

Definition (Homogeneous Patterns)

For each level of the corresponding tree the inner nodes have to be labeled with the same operation. The *type* is the sequence of operators on the path from the root to the deepest leaf.

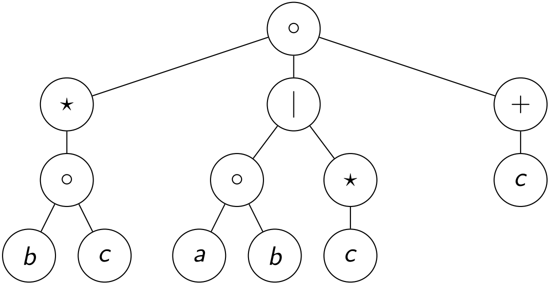


$[(abc \mid c)(a \mid dc)c(db \mid c \mid bd)]^+$:
Homogeneous pattern of type $+\circ|\circ$ and depth 4.

Homogeneous Patterns – Example II

Definition (Homogeneous Patterns)

For each level of the corresponding tree the inner nodes have to be labeled with the same operation. The *type* is the sequence of operators on the path from the root to the deepest leaf.



$(bc)^*(ab \mid c^*)c^+$: Not a homogeneous pattern!

Current Results

General Patterns
$\bar{\mathcal{O}}(nm/\log^{3/2} n)$ and $\Omega(nm/\log^{7+\epsilon} n)$, assuming FSH
Homogeneous Patterns
Several common problems need only homogeneous patterns and are solvable in time $\mathcal{O}(n \log^2 m + m)$ (e.g. dictionary, superset and string matching).

Dichotomy for homogeneous types [BI16], [BGL17]

- It suffices to analyse few pattern types of constant depth
- For "easy" patterns: strongly sub-quadratic time algorithms
- For "hard" patterns: $\Omega((nm)^{1-\epsilon})$ lower bound, assuming SETH

Questions: Does the general lower bound transfer to the hard patterns? Are there super-poly-logarithmic improvements as for APSP and OV?

Our Results

Before:
In general: $\bar{\mathcal{O}}(nm/\log^{3/2} n)$
For "hard" homogeneous patterns: $\Omega((nm)^{1-\epsilon})$, assuming SETH.
New bounds assuming FSH:

<table>
<tr><th></th><th>$\circ{+}|$, $\circ|{+}$, $\circ{+}\circ$, $\circ|\circ$, $\circ\star$</th><th>$|{+}|\circ$</th><th>$|\circ|$, $|\circ{+}$</th><th>$+|\circ|$, $+|\circ{+}$</th></tr>
<tr><td>Pattern matching</td><td rowspan="2">$\Theta\left(\frac{nm}{\text{poly}\log n}\right)$</td><td>$\Theta(n+m)$</td><td>$\frac{nm}{2^{\Omega(\sqrt{\log n})}}$</td><td>same as $|\circ|$, $|\circ{+}$</td></tr>
<tr><td>Membership</td><td>$\Theta\left(\frac{nm}{\text{poly}\log n}\right)$</td><td>$\Theta(n+m)$</td><td>$\frac{nm}{2^{\Omega(\sqrt{\log n})}}$</td></tr>
</table>

- $2^{\Omega(\sqrt{\log n})} \in \omega(\text{poly}\log n)$:
  Currently fastest algorithm and best we can hope for under SETH.
- For "ultra-hard" pattern types: The general algorithm is optimal up to a constant number of log-factors.

Tight dichotomy for homogeneous pattern types (up to log-factors).

Upper Bounds

The Polynomial Method

- Originally used for circuit lower bounds (Razborov 1987 and Smolensky 1987)
- Method to transform boolean circuits into polynomials
- Adopted by Williams in 2014 to improve algorithms for APSP
- Yields super-poly-logarithmic runtime improvements
- Idea: Solve the task for many small sub-problems in parallel

Orthogonal Vectors (OV)

Input: Sets $U, V \subseteq \{0,1\}^d$ of n vectors each.
Question: Are there $u \in U, v \in V$ such that $\langle u, v \rangle = 0$?

Lemma (Chan and Williams 2016)

For $d = 2^{\Theta(\sqrt{\log n})}$ OV can be solved in time $n^2/2^{\Omega(\sqrt{\log n})}$ deterministically.

Fast Algorithm

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Focus on patterns with type $|\circ|$: $[(a \mid b)(b \mid c)d] \mid [ab(a \mid c \mid d)] \mid [bd]$
Main observation: Patterns can be split into independent sub-patterns!

- Define a threshold $f \in 2^{\Theta(\sqrt{\log n})}$
- Split p into *large* (matching $> f$ symbols) and *small* sub-patterns
- Solve each of the $\leq k = m/f$ large sub-patterns of type $\circ|$ with the known near-linear time algorithm:

$$\mathcal{O}\left(\sum_{i=1}^{k} n \log^2 m_i + m_i\right) = \mathcal{O}\left(\frac{m}{f} n \log^2 n + m\right) = \frac{nm}{2^{\Omega(\sqrt{\log n})}}$$

- Reduce small sub-patterns to OV with dimension $d = 2^{\Theta(\sqrt{\log n})}$:

$$\frac{nm}{2^{\Omega(\sqrt{\log n})}}$$

Small Sub-Patterns

Assume w.l.o.g. that all sub-patterns match exactly f symbols.

Check whether there is a sub-pattern q and an offset $i \in [n]$ such that:

$$
\begin{array}{c}
t_1 t_2 \ldots\ldots t_i t_{i+1} \ldots t_{i+f-1} \ldots\ldots t_{n-1} t_n \\
q_1 q_2 \;\ldots\; q_f
\end{array}
$$

- Use all length f substrings of t as one set of vectors
- Use sub-patterns as the other set of vectors
- Encode orthogonality using characteristic vector for Σ:

$$
\begin{array}{l} t_i = a \\ q_j = a \mid b \end{array} \longrightarrow \begin{array}{c} (1,0,0) \\ (1,1,0) \end{array} \longrightarrow \begin{array}{c} (1,0,0) \\ (0,0,1) \end{array}
$$

$$\Sigma = \{a, b, c\}$$

- n text-vectors and $\leq m$ pattern-vectors
- dimension $d = f \cdot |\Sigma| \in 2^{\Theta(\sqrt{\log n})}$ if $|\Sigma| \in 2^{\mathcal{O}(\sqrt{\log n})}$
- Use Bloom-Filters for larger alphabets to keep dimension small

Lower Bounds

Satisfiability Problems

SETH rules out **polynomial** improvements.
We want to rule out **log-factor** improvements!
$\rightarrow$ We need a stronger assumption!

Monotone De Morgan Formula

A node labeled tree. Each inner node is labeled with AND or OR. Each leaf is labeled with a variable. Size = number of leaves.

Formula-Pair (Abboud and Bringmann 2018)

Input: A monotone De Morgan formula F with s inputs, each input is used exactly once, and sets $A, B \subseteq \{0,1\}^{s/2}$ of size n and m.
Task: Check whether there are $a \in A$, $b \in B$ such that $F(a, b) = \text{true}$.

Formula-Pair Hypothesis (FPH)

For all $k \geq 1$: For a monotone De Morgan formula F of size s and sets $A, B \subseteq \{0,1\}^{s/2}$ of n half-assignments each, Formula-Pair cannot be solved in time $\mathcal{O}(n^2 s^k / \log^{3k+2} n)$, in the Word-RAM model.

General Idea

Reduce **Formula-Pair** to **pattern matching** with a text t and pattern p of a specific type:

$$(\exists a \in A, b \in B : F(a, b) = \text{true}) \iff t \in \mathcal{M}(p)$$

We first encode the formula such that for all $a \in A, b \in B$:

$$F(a, b) = \text{true} \iff t(a) \in \mathcal{L}(p(b))$$

Encode the INPUT, AND and OR gates of the formula inductively.

Focus on patterns of type $\circ{+}\circ$: $ab(bc)^+b^+(cd)^+$

Encoding the Formula I

INPUT Gate $F_g(a,b) = a_i$

$$t_g := 0 a_i 1 \qquad u_g := 0011$$
$$p_g := 0^+ 1 1^+ \qquad q_g := 0^+ 1^+$$

if $a_i = 1$

For $F_g(a,b) = b_i$ define: $t_g := 011 \quad p_g := 0^+ b_i 1^+$

AND Gate $F_g(a,b) = F_{g_1}(a,b) \wedge F_{g_2}(a,b)$

$$t_g := t_1 G t_2 \qquad u_g := u_1 G u_2$$
$$p_g := p_1 G p_2 \qquad q_g := q_1 G q_2$$

if $t_i \in \mathcal{L}(p_i)$
for $i = 1, 2$

Define separator gadget $G := 2\langle g \rangle 2$.
Need universal text u_g and pattern q_g for OR gate.

Encoding the Formula II

OR Gate $F_g(a,b) = F_{g_1}(a,b) \vee F_{g_2}(a,b)$

$$
\begin{aligned}
t_g &:= (u_1 G u_2) G (u_1 G u_2) G (\mathbf{t_1} G \mathbf{t_2}) G (u_1 G u_2) G (u_1 G u_2) \\
p_g &:= (u_1 G u_2 G)^+ (\mathbf{p_1} G q_2) G (q_1 G \mathbf{p_2}) (G u_1 G u_2)^+ \\
u_g &:= (u_1 G u_2) G (u_1 G u_2) G (\mathbf{u_1} G \mathbf{u_2}) G (u_1 G u_2) G (u_1 G u_2) \\
q_g &:= (u_1 G u_2) G (u_1 G u_2) G (\mathbf{q_1} G \mathbf{q_2}) G (u_1 G u_2) G (u_1 G u_2)
\end{aligned}
$$

Lemma (Size bound for the encoding)

$|u_r|, |t_r|, |p_r|, |q_r| \in \mathcal{O}(5^{d(F)} s \log s)$, *with* r *as root of* F.

Proof.

$|p_g| \in \mathcal{O}(|u_g|) = \mathcal{O}(|t_g|) = \mathcal{O}(|q_g|)$.
By definition: $|u_g| \leq 5|u_1| + 5|u_2| + \mathcal{O}(\log s)$
Inductively over the $d(F_g)$ levels of F_g: $|u_g| \leq \mathcal{O}(5^{d(F_g)} s \log s)$. □

Outer OR

Formula-Pair

Task: Check whether $\exists a \in A, b \in B : F(a,b) = \text{true}$.

$$t := \bigodot_{a \in A} (33u_r3u_r3u_r3t(a)3u_r3u_r3u_r3u_r)$$

$$p := 3u_r3u_r3u_r3u_r \bigodot_{b \in B} (3^+(u_r3)^+u_r3^+q_r3p(b)3(u_r3)^+q_r)3u_r3u_r3u_r3u_r$$

Lemma (Correctness)

$(\exists a \in A, b \in B : F(a,b) = true) \iff t \in \mathcal{M}(p)$

The Lower Bound

- Text length and pattern size is $\mathcal{O}(n5^d s \log s)$.
- We can reduce the depth of a formula by increasing its size: $d \to d' = \mathcal{O}(\log s)$ and $s \to s' = \mathcal{O}(s^2)$. (e.g. Bonet and Buss 1994)
- The final text length and the pattern size is $\mathcal{O}(ns^{15})$.
- Assume a $\mathcal{O}(NM/\log^{92} N)$ algorithm for $\circ{+}\circ$-pattern matching:

$$\mathcal{O}\left(\frac{ns^{15} \cdot ns^{15}}{\log^{92}(ns^{15})}\right) \subseteq \mathcal{O}\left(\frac{n^2 s^{30}}{\log^{92} n}\right)$$

Formula-Pair Hypothesis (FPH)

For all $k \geq 1$: For a monotone De Morgan formula F of size s and sets $A, B \subseteq \{0,1\}^{s/2}$ of n half-assignments each, Formula-Pair cannot be solved in time $\mathcal{O}(n^2 s^k / \log^{3k+2} n)$, in the Word-RAM model.

Conclusion

Conclusion

Before: Upper bound: $\bar{\mathcal{O}}(nm/\log^{3/2} n)$
Lower bound: $\Omega((nm)^{1-\epsilon})$, assuming SETH.

New bounds assuming FPH	$\circ{+}\vert$, $\circ\vert{+}$, $\circ{+}\circ$, $\circ\vert\circ$, $\circ\star$	$\vert{+}\vert\circ$	$\vert\circ\vert$, $\vert\circ{+}$	${+}\vert\circ\vert$, ${+}\vert\circ{+}$
Pattern matching	$\Theta\left(\frac{nm}{\text{poly}\log n}\right)$	$\Theta(n+m)$	$\frac{nm}{2^{\Omega(\sqrt{\log n})}}$	same as $\vert\circ\vert$, $\vert\circ{+}$
Membership		$\Theta\left(\frac{nm}{\text{poly}\log n}\right)$	$\Theta(n+m)$	$\frac{nm}{2^{\Omega(\sqrt{\log n})}}$

$\rightarrow$ A tight dichotomy for the hard pattern types.

Additional Material

Satisfiability Problems I

De Morgan Formula

A node labeled tree. Each inner node is labeled with AND or OR. Each leaf is labeled with a variable or its negation. Size = number of leaves

Formula-SAT (Abboud and Bringmann 2018)

Input: A De Morgan formula F of size s on n variables.
Task: Check whether there is a satisfying assignment for F.

Formula-SAT Hypothesis (FSH) (Abboud and Bringmann 2018)

Formula-SAT on De Morgan formulas of size $s = n^{3+\Omega(1)}$ cannot be solved in $\mathcal{O}(2^n/n^\epsilon)$ time, for some $\epsilon > 0$, in the Word-RAM model.

FSH also used as hypothesis for the lower bound for the general case.

Satisfiability Problems II

Formula-Pair (Abboud and Bringmann 2018)

Input: A *monotone* De Morgan formula F with s inputs, each input is used exactly once, and sets $A, B \subseteq \{0,1\}^{s/2}$ of size n and m, respectively.
Task: Check whether there are $a \in A$, $b \in B$ such that $F(a, b) = \text{true}$.

Formula-SAT reduces to Formula-Pair by writing down all half-assignments explicitly. We can also ensure that each input is used exactly once.

Formula-Pair Hypothesis (FPH)

For all $k \geq 1$: For a monotone De Morgan formula F of size s and sets $A, B \subseteq \{0,1\}^{s/2}$ of n half-assignments each, Formula-Pair cannot be solved in time $\mathcal{O}(n^2 s^k / \log^{3k+2} n)$, in the Word-RAM model.

Reducing Pattern Matching to Membership

We can reduce pattern matching to membership for the pattern types for which we showed improved lower bounds.

Example for patterns of type $\circ{+}|$:

- $t' := \sigma t \sigma$ where $\sigma \in \Sigma$
- $p' := \Sigma^+ p \Sigma^+ = (\sigma_1 \mid \sigma_2 \mid \ldots \mid \sigma_{\mathfrak{s}})^+ p (\sigma_1 \mid \sigma_2 \mid \ldots \mid \sigma_{\mathfrak{s}})^+$

$t \in \mathcal{M}(p) \iff t' \in \mathcal{L}(p')$

"$\Rightarrow$" The first Σ^+ matches the initial σ and the not matched prefix of t. Analogous for the second Σ^+.
"$\Leftarrow$" The two Σ^+ match at least the initial and final σ. Thus, p has to match some substring of t (possibly the empty string).